Sutured manifolds, L^2-Betti numbers and an upper bound on the leading coefficient

Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften (Dr. rer. nat.) der Fakultät Mathematik der Universität Regensburg

vorgelegt von

Gerrit Herrmann aus

Köln im Jahr

2019

Promotionsgesuch eingereicht am:

Die Arbeit wurde angeleitet von Prof. Stefan Friedl.

Contents

Part I.

Prelude

Introduction

1. Motivation

A knot K in S^3 is a smooth embedding $K\colon S^1 \to S^3$. It is a classical result due to Seifert [Sei35] that for every knot K there exists an embedding $i\colon F \to S^3$ of a connected compact oriented surface F such that $i|_{\partial F} = K$. Such an embedded surface is called a Seifert surface for K. Once the existence of such a surface is established, there are three natural and interesting questions to consider:

Question 0.1. When does a Seifert surface for K have minimal genus compared to all other Seifert surfaces for K?

Question 0.2. How are two minimal genus Seifert surfaces for the same knot related?

Question 0.3. Can we use minimal genus Seifert surfaces to distinguish knots?

A very simple example is the unknot. The unknot bounds a disc which has genus zero. Since there is no lower genus, the disc is a minimal genus Seifert surface for the unknot. It is also a classical fact that all smooth embeddings of D^2 into S^3 are isotopic. There is only one isotopy class of a minimal genus Seifert surface for the unknot. Since the unknot is the only knot to bound a disc, we see that the minimal Seifert surface helps us to distinguish the unknot from other knots. Finding answers to the above questions becomes much more challenging if one considers more difficult knots.

Before we look at the questions more closely, we describe a generalisation of the concept of minimal genus Seifert surface to arbitrary 3-manifolds due to Thurston. He considered for a surface S the complexity defined by $\chi_-(S) := -\chi(S \setminus {}^{\text{discs and}}_{\text{spheres}})$. Given a 3-manifold N, Thurston [Th86] studied the value

$$\|\sigma\| := \min\{\chi_-(S) \mid S \subset N \text{ properly embedded surface and } [S] = [\sigma]\}$$

for a homology class $\sigma \in H_2(N, \partial N; \mathbb{Z})$. Nowadays one refers to the value as the Thurston norm of σ.

The three questions above can be generalised as follows.

Question 0.4. When is a given surface S in N Thurston norm minimizing?

Question 0.5. How are two Thurston norm minimizing surfaces for the same homology class related?

Question 0.6. Can we use Thurston norm minimizing surfaces to distinguish 3-manifolds?

The author contributes to all three questions some results in this thesis.

Characterisation of Thurston norm minimizing surfaces

For technical reasons we restrict Question 0.4 to the following subclass of surfaces. A decomposition surface S in a 3-manifold N is a surface without discs and spheres and for any choice of base point $x \in S$ the inclusion $S \to N$ induces a monomorphism $\pi_1(S, x) \to \pi_1(N, x)$.

Friedl–T. Kim showed a characterisation of Thurston norm minimizing surfaces in terms of twisted homology groups:

Theorem 0.7 ([FK13, Theorem 1.1] **).** *Let S be a non-separating non-empty decomposition surface in a connected compact oriented 3-manifold N with empty or toroidal boundary. We abbreviate $M = N \setminus S \times (-1, 1)$ and $S_- = S \times \{-1\}$. The surface S has minimal complexity in its homology class if and only if there exists a unitary representation $\alpha\colon \pi_1(M) \to U(k)$ such that $H_*(M, S_-; \mathbb{C}^k_\alpha) = 0$.*

The existence proof of the finite dimensional representation is not explicit and one is stuck with the problem of finding such a representation. However, every group G has a canonical representation into the isometries of $L^2(G)$. If G is not finite, then these representations are not finite dimensional. But there is a nice dimension function due to von Neumann. Following the idea of Atiyah [At76] one can define the L^2-Betti numbers $b_*^{(2)}(X)$ of a space X using the von Neumann dimension.

In this thesis we prove an L^2-analogue of the result of Friedl and T. Kim.

Theorem 0.8. *Let S be a non-empty decomposition surface in a connected compact oriented 3-manifold N with empty or toroidal boundary. We abbreviate $M = N \setminus S \times (-1, 1)$ and $S_- = S \times \{-1\}$. The surface S has minimal complexity in its homology class if and only if $b_*^{(2)}(M, S_-) = 0$.*

See Theorem 3.1 for a more general statement. The proof of the theorem uses the virtual fibring theorem of Agol [Ag08].

New invariants from Thurston norm minimizing surfaces

In the last 40 years an extensive theory has been built around L^2-Betti numbers. One part of this theory is the L^2-torsion $\tau^{(2)}(X, Y)$ which associates to a finite CW-pair (X, Y) a positive real number. If all L^2-Betti numbers for the pair (X, Y) vanish, then $\tau^{(2)}(X, Y)$ is defined under some technical extra assumptions. So as a side product of the main theorem we obtain a new invariant $\tau^{(2)}(N \setminus \nu(\Sigma), \Sigma_-)$ for the pair (N, Σ), where N is a 3-manifold and Σ is a Thurston norm minimizing decomposition surface. Here $\nu(\Sigma)$ denotes an open tubular neighbourhood.

In this thesis we show the following contribution to Question 0.5 which generalises the work of Kakimzu [Ka92] who proved a similar statement for link exteriors.

Theorem 0.9. *If S and T are Thurston norm minimizing decomposition surfaces in a connected compact oriented irreducible 3-manifold N with empty or toroidal boundary and* *such that* $[S] = [T] \in H_2(N, \partial N; \mathbb{Z})$*, then there is a sequence of Thurston norm minimizing decomposition surfaces* $S_1, \ldots, S_n$ *such that* $S_1 = S$*,* $S_n = T$*,* $S_i \cap S_{i+1} = \emptyset$*, and* $[S_i] = [S] = [T] \in H_2(N, \partial N; \mathbb{Z})$ *for all* $i \in \{1, \ldots, n\}$*.*

We use this theorem to show that the value $\tau^{(2)}(N \setminus \nu(\Sigma), \Sigma_-)$ is in fact an invariant of the pair (N, σ) and does not depend on the choice of Thurston norm minimizing decomposition surface representing $\sigma \in H_2(N, \partial N; \mathbb{Z})$:

Theorem 0.10. *Let* N *be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary. If* S *and* T *are two Thurston norm minimizing decomposition surfaces for the same class i.e.* $[S] = [T] \in H_2(N, \partial N; \mathbb{Z})$*, then we have*

$$\tau^{(2)}(N \setminus \nu(S), S_-) = \tau^{(2)}(N \setminus \nu(T), T_-).$$

This theorem can also be seen as a contribution to Question 0.6.

Comparison to other invariants

The last important result of this thesis compares the value $\tau^{(2)}(N \setminus \nu(S), S_-)$ to another invariant. In 1928 Alexander [Al28] associated to a knot K a polynomial which today is called the Alexander polynomial of K. Several years ago an L^2-analogue of the Alexander polynomial, the L^2-Alexander torsion, has been introduced and has been an object of intensive investigation ever since.

After normalisation the L^2-Alexander torsion associates to a 3-manifold N and a cohomology class $\phi \in H^1(N; \mathbb{Z})$ a function $\tau^{(2)}(N, \phi)\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. Liu [Li17] showed that there is a real number $C(N, \phi) \in \mathbb{R}_{>0}$ such that the L^2-Alexander torsion $\tau^{(2)}(N, \phi)$ has the following asymptotic behaviour:

$$\lim_{t \to 0} \tau^{(2)}(N, \phi)(t) = C(N, \phi) \text{ and } \lim_{t \to \infty} \tau^{(2)}(N, \phi)(t) \cdot t^{-\|\phi\|} = C(N, \phi).$$

We refer to $C(N, \phi)$ as the leading coefficient.

In this thesis we show that the relative torsion is an upper bound to the leading coefficient. This is a joint work with Ben-Aribi and Friedl.

Theorem 0.11 ([BFH18])**.** *Let* N *be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary. For any non-zero* $\phi \in H^1(N; \mathbb{Z})$ *and any* Σ *which is a Thurston norm minimizing decomposition surface dual to* ϕ *the following inequality holds:*

$$C(N, \phi) \leqslant \tau^{(2)}(N \setminus \nu(\Sigma), \Sigma_-).$$

2. Organisation of the thesis

In Chapter 1 we present the basic results about Thurston norm and Thurston norm minimizing surfaces. Most of the results are very classical and well known. We state them mostly to fix our notations and conventions. The only new result is

Proposition 1.42 which was certainly known to the experts but could not be found in the literature. In Chapter 2 we introduce L^2-invariants and summarise the result we later need in this thesis. Chapter 3 contains the proof of Theorem 0.8.

From the results in Chapter 3 we obtain a new invariant $\tau^{(2)}(N \setminus \nu(\Sigma), \Sigma_-)$ for the pair (N, Σ), where N is a 3-manifold and Σ is a Thurston norm minimizing surface. In Chapter 4 we study its basic properties and show that it is actually an invariant for the pair (N, ϕ), where $\phi \in H^1(N; \mathbb{Z})$ is the Poincaré dual of $[\Sigma] \in H_2(N, \partial N; \mathbb{Z})$.

In Chapter 5 we introduce the leading coefficient $C(N, \phi)$ and compare it to the relative L^2-torsion $\tau^{(2)}(N \setminus \nu(\Sigma), \Sigma_-)$. The main result in Chapter 5 is Theorem 0.11.

In the last chapter we provide some calculations for $\tau^{(2)}(N \setminus \nu(\Sigma), \Sigma_-)$. These calculations lead to some open questions which we briefly discuss.

3. Acknowledgements

I would like to thank Fritz Ostermann for introducing me to higher mathematics. I am grateful to Amelie Bönniger for all the love and that she is always by my side to support me.

A big debt of gratitude is owed to my advisor Stefan Friedl. He explained to me many beautiful results in geometric topology. I enjoyed hundreds of hours of discussions about various questions and problems in algebraic topology which arose in the preparation of his courses. I admire his passion for teaching.

I can only be thankful for how much support I received from Friedrich Herrmann and Marianne Ostermann over the last ten years. I would not be able to pursue my goals without these two amazing people.

During my PhD, I was partially funded by the SFB 1085, which is funded by the Deutsche Forschungsgemeinschaft.

A brief history of 3-manifold topology

Here we take a quick tour through the history of 3-manifolds. This exposition is far from being complete and the choice of theorems presented here heavily depends on the preference of the author. For example we ignore minimal surface theory or Heegaard Floer homology which are doubtlessly very important subjects in 3-dimensional topology. For simplicity we restrict ourself to closed connected orientable 3-manifolds unless explicitly said otherwise. In order to stay short and to focus on the narrative, we are often mathematically imprecise.

One of the first treatments of 3-manifolds is given by Poincaré in the series of five papers „Analysis situs". There Poincaré gives two versions of what today is called Poincaré conjecture.

1. The 3-sphere is the only 3-manifold with vanishing first Betti and torsion number.

2. The 3-sphere is the only 3-manifold with trivial fundamental group.

While Poincaré himself noticed that the first conjecture is wrong, the second conjecture became a leading force behind the developments in low dimensional topology till Perelman proved it more than 100 years later.

A first result in the direction of classifying all 3-manifolds is due to Herbert Seifert in his work *"Topologie dreidimensionaler gefaserter Räume"*. He defines what today is called a Seifert fibre space. In modern language a Seifert fibre space is a 3-manifold which admits an S^1-bundle structure over a 2-dimensional orbifold. He proved the following result.

Theorem 0.1 (Seifert [Sei33] (1933))**.** *The homeomorphism type of a Seifert fibre space can be read off from the base orbifold and the Euler class of the bundle.*

Here the expression „read off" means that a 3-manifold M may fibre over an orbifold in different ways, but all these different ways are completely understood. Moreover, there is a classification of 2-dimensional orbifolds. Going through the classification shows that the Poincaré conjecture is true within the class of Seifert fibred spaces.

A result which gives in general a better understanding of the structures of 3-manifolds is due to Kneser and Milnor.

Theorem 0.2 (Kneser [Kne29] (1929) , Milnor [Mi62a] (1962))**.** *Let M be a 3-manifold. There is a decomposition*

$$M = S^3 \# P_1 \# \ldots \# P_n,$$

where each P_i is prime. This decomposition is unique up to permutation of the summands.

The existence part was proven by Kneser and the uniqueness part was proven by Milnor. Note that nowadays it is known that such a prime decomposition exists in all dimensions but the uniqueness is special for dimension 3 and smaller. The modern treatment uses that the Poincaré conjecture is proven in any dimension. But Kneser's proof in dimension 3 does not rely on it. He worked with an additional assumption, namely that M is triangulated. When Milnor proved the uniqueness part of the decomposition it was already known, that all 3-manifolds can be triangulated. To be more precise Moise proved the following:

Theorem 0.3 (Moise [Mo52] (1952)). *Every 3-manifold admits a triangulation and this triangulation is unique up to subdivision.*

Five years after Moise the Greek mathematician Papakyriakopoulos proved two further fundamental results.

Theorem 0.4 (Loop theorem, Papakyriakopoulos [Pa57] (1957)). *Let M be a 3-manifold with non-empty boundary and $\ker(\pi_1(\partial M) \to \pi_1(M)) \neq \{1\}$. There is a properly embedded disc D^2 such that $[\partial D^2] \neq 0 \in \pi_1(\partial M)$.*

The theorem was first conjectured by Dehn in 1910. Dehn famously gave a wrong proof. The gap in Dehn's proof was first discovered in 1929 by Kneser. A very useful corollary relates geometric and algebraic properties of surfaces in 3-manifolds. An embedded surface S in a 3-manifold M is called two-sided if S has a trivial normal bundle and is called compressible if there is a disc D in M such that $S \cap D = \partial D$ is an essential simple closed curve on S. From the loop theorem one can deduce the following equivalence: An embedded surface S in M is compressible if and only if $\ker(i_*\colon \pi_1(S) \to \pi_1(M)) \neq \{1\}$.

A cousin of the loop theorem is the sphere theorem.

Theorem 0.5 (Sphere theorem, Papakyriakopoulos [Pa57] (1957)). *Let M be a 3-manifold with $\pi_2(M) \neq 0$. Then there exists an embedding $g\colon S^2 \to M$ with the property that $[g] \neq 0 \in \pi_2(M)$.*

A 3-manifold M is called irreducible if every embedding $f\colon S^2 \to M$ extends to an embedding $\overline{f}\colon D^3 \to M$. Obviously an irreducible 3-manifold is prime. The converse is also true except for $S^2 \times S^1$. As a consequence of the sphere theorem one has that an irreducible 3-manifold M has trivial π_2. Moreover, if M has an infinite fundamental group, then by applying Hurewicz to the universal cover one sees that all other homotopy groups are trivial, too. Hence most irreducible 3-manifolds are so called Eilenberg-Maclane spaces and all homotopy properties are encoded in the fundamental group.

We call a surface S with infinite fundamental group in M incompressible, if $\pi_1(S) \to \pi_1(M)$ is a monomorphism for every choice of base point. The presence of an incompressible surface in a 3-manifold is very useful and was first exploited by the German mathematician Haken. Therefore, an irreducible 3-manifold M is called Haken if M admits a two-sided incompressible surface. Haken proved the following algorithmic classification of Haken 3-manifolds.

Theorem 0.6 (Haken [Ha62] (1962))**.** *There exists an algorithm that decides whether two triangulated Haken 3-manifolds are homeomorphic.*

The idea of his proof was so influential on the theory of 3-manifolds that we outline it here roughly. If S is an incompressible surface in an irreducible 3-manifold M, then one can remove a tubular neighbourhood of S in M to obtain a 3-manifold M_1 with boundary. It turns out that M_1 again contains a two-sided incompressible surface S_1 and one can remove a tubular neighbourhood of S_1. This process can be repeated over and over again, till one ends up with the disjoint union of 3-balls. Haken proved that this construction under some mild technical assumptions on the S_i's terminates after finitely many steps. This enables induction arguments in the world of 3-manifolds by first proving a statement for the 3-ball and then by showing that the statement is preserved after gluing a long an incompressible surface.

Waldhausen used the same technique to prove the following topological rigidity.

Theorem 0.7 (Waldhausen [Wa68] (1968))**.** *If an irreducible 3-manifold M admits a two sided incompressible surface, then every homotopy-equivalence is homotopic to a homeomorphism.*

Not every 3-manifold is Haken. For example the so called Weeks manifold is not Haken. However Waldhausen's techniques worked for the case that M has a finite cover, which is Haken. So the question arose whether every 3-manifold with infinite fundamental group is finitely covered by a Haken 3-manifold.

Extending earlier work of Waldhausen the mathematicians Jaco and Shalen and independently Johannson proved the following decomposition result, which is now known as the JSJ-decomposition. A 3-manifold M with torus boundary is called atoroidal if every incompressible torus in M is isotopic to a boundary component.

Theorem 0.8 (Jaco, Shalen [JS76] (1976), Johannson [Jo75] (1975))**.** *Let M be an irreducible 3-manifold. There is a collection of disjointly embedded tori $T_1, \ldots, T_n$ such that M cut along the tori is the disjoint union of 3-manifolds which are either Seifert fibred spaces or atoroidal.*

Thurston observed that the topological condition of atoroidal is sufficient to get a rich geometric structure.

Theorem 0.9 (Thurston [Th86a], [Th86b], [Th86c] (1986))**.** *If M is an atoroidal Haken manifold with infinite fundamental group, then M is hyperbolic.*

Thurston's proof distinguishes two cases: Either M fibres over the circle or it does not. The second case was proven again using Haken hierarchies. He was wondering if the second case is really necessary and asked the following question: Admits any atoroidal 3-manifold with infinite fundamental group a finite cover which fibres over the circle?

In the years between 2002 and 2004 Perelman made an important improvement on Thurston's theorem by dropping the assumption that the 3-manifold is Haken. He proved using Ricci-flow techniques:

Theorem 0.10 (Perelman [Pe02], [Pe03] (2002)). *If M is an irreducible atoroidal 3-manifold with infinite fundamental group, then M is hyperbolic.*

In these papers Perelman proved much more. He proved that every 3-manifold with finite fundamental group is a quotient of S^3 by an action of a finite subgroup of $SO(4)$. In particular, he proved the Poincaré conjecture.

Theorem 0.11. *If M is a 3-manifold with $\pi_1(M) = 0$, then $M \cong S^3$.*

While Poincaré's original conjecture was proven, two interesting questions were still open. Namely whether every irreducible 3-manifold with infinite fundamental group is finitely covered by a Haken manifold or even stronger by a surface bundle over S^1. These two conjectures were proven in 2008 by Agol building upon the work of Wise.

Theorem 0.12 (Agol [Ag08] (2008), Wise (2008)). *If M is an irreducible 3-manifold such that the JSJ-decomposition contains at least one hyperbolic piece, then M admits a finite cover $\widehat{M}$ which fibres over the circle.*

Part II.

Main Part

CHAPTER 1

Surfaces in 3-manifolds

1.1. Thurston norm

2nd homology and surfaces

By a surface we mean an oriented 2-dimensional manifold not necessarily connected. A surface also can have boundary. In the study of the second homology surfaces play an important role. This is especially true for 3-manifolds. However, we start with a rather general lemma.

Lemma 1.1. *Let X be a topological space. For any cycle $\sigma = \sum_{i=1}^{n} n_i \cdot \sigma_i \in C_2(X;\mathbb{Z})$ there exists a surface Σ and a map $f\colon \Sigma \to X$ such that $f_*([\Sigma]) = [\sigma]$ and $-\chi(\Sigma) \leqslant \frac{1}{2}\sum_{i=1}^{n} |n_i|$.*

Proof . With out loss of generality, we can write σ as a sum of simplexes such that each coefficient is equal to 1 i. e. $\sigma = \sum_{i=1}^{m} \sigma_i$.

Since σ is a cycle, we have $\partial\sigma = \sum_{i=1}^{m} \partial\sigma_i = 0$. Therefore, every face e in the sum $\sum_{i=1}^{m} \partial\sigma_i$ appears in an even number but with opposite signs. We obtain a pairing of the edges of the σ_i. We can build a topological manifold (not necessarily connected) Σ by taking a 2-simplex for each σ_i and glue the edges along the chosen pairing. One easily verifies that the resulting topological space is a manifold. Moreover, the maps from the simplexes pass down to the quotient and hence describe a map $f\colon \Sigma \to X$ with $f_*([\Sigma]) = [\sigma]$.

The surface Σ has by construction $\sum_{i=1}^{n} |n_i|$ many 2-cells and $\frac{3}{2}\sum_{i=1}^{n} |n_i|$ many 1-cells. It also has more than zero 0-cells so that we have

$$\chi(\Sigma) \geqslant \sum_{i=1}^{n} |n_i| - \frac{3}{2}\sum_{i=1}^{n} |n_i| = -\frac{1}{2}\sum_{i=1}^{n} |n_i|. \qquad \square$$

In the case that X is a compact 3-manifold, we can strengthen this result to the following theorem.

Theorem 1.2. *If M is a compact oriented 3-manifold and $A \subset \partial M$ a closed codimension 0 submanifold, then every class $\sigma \in H_2(M, A;\mathbb{Z})$ can be represented by a properly embedded surface Σ i. e. there is an proper embedding $i\colon (\Sigma, \partial\Sigma) \to (M, A)$ such that $i_*([\Sigma, \partial\Sigma]) = \sigma$.*

For the proof of this statement we need the following construction.

Figure 1.1.: This figure shows the surgery which replaces $c \times X$ by $c \times I \sqcup c \times I$.

Definition 1.3 (Oriented sum)**.** Let S and T be two properly embedded surfaces in an oriented 3-manifold M which are in general position. Then the intersection $c = T \cap S$ is the disjoint union of compact 1-manifolds. Let V be a tubular neighbourhood of c. If V is small enough, then the intersection of V with $T \cup S$ is topologically $c \times X$. Here X is the topological space which looks like the Latin letter X. Now there is a unique way to replace X by two disjoint intervals with the same endpoints and matching orientations (see Figure 1.1). We refer to the resulting surface by $T \oplus S$. One easily sees that $[T \oplus S] = [T] + [S]$ holds in homology.

Proof of Theorem 1.2 . The proof has some bootstrap logic. We start with the case $A = \partial M$ and from there we extend to general A. The trivial class is represented by the empty manifold. Therefore, let $\sigma \in H_2(M, \partial M; \mathbb{Z})$ be non-zero. By Poincaré duality and by the fact that S^1 is an Eilenberg-Maclane space of type $K(\mathbb{Z}, 1)$ we have

$$H_2(M, \partial M; \mathbb{Z}) \cong H^1(M; \mathbb{Z}) \cong [M, S^1].$$

Since M and S^1 are admitting a smooth structure we can use smooth approximation to obtain a smooth map $f\colon M \to S^1$, which under the above isomorphisms represents the class σ i.e. $D_M(\sigma) = f^*(\theta)$, where $\theta \in H^1(S^1; \mathbb{Z})$ is the standard generator and D_M is Poincaré duality in M.

By Sard's theorem we find a regular value $y \in S^1$ and by the implicit function theorem we have that $\Sigma := f^{-1}(\{y\})$ is a submanifold of M.

Claim. We have the equality $[\Sigma] = \sigma$ in $H_2(M, \partial M; \mathbb{Z})$.

First we observe that being a regular value is an open condition and we find a small neighbourhood $V \subset S^1$ of y. We obtain an open neighbourhood $U := f^{-1}(V)$. By shrinking V and U we can assume that U is a tubular neighbourhood of Σ and V is a tubular neighbourhood of y (viewed as a codimension 1 submanifold of S^1) and $f\colon U \to V$ is a bundle map.

The generator $\theta \in H^1(S^1; \mathbb{Z})$ is the Thom class of y viewed as a submanifold of S^1. The Thom class is compatible with bundle maps and hence $f^*(\theta) \in H^1(M; \mathbb{Z})$ is the Thom class of Σ viewed as a submanifold of M. By definition the Thom class is given by $D_M([\Sigma])$. By the construction of f we have that $f^*(\theta) = D_M(\sigma)$ and hence $[\Sigma] = \sigma$ as claimed. We refer to Bredon's book [Br93, Chapter VI Section 11] for more details about the Thom class.

We just proved the case $A = \partial M$. Next we consider a general subsurface $A \subset \partial M$. We obtain surfaces $S_1, \dots, S_n$ by cutting ∂M along the interior of A i. e.

$$\partial M \setminus A^o = S_1 \cup \dots \cup S_n.$$

By excision one sees that $\{[S_1], \dots [S_n]\}$ is a generating set for $H_2(\partial M, A; \mathbb{Z})$. Hence every element can be represented by an embedded surface in a collar neighbourhood of ∂M in M. The collar neighbourhood is necessary to deal with multiples of $[S_i]$. We have the long exact sequence in homology:

$$H_2(\partial M, A; \mathbb{Z}) \xrightarrow{i} H_2(M, A; \mathbb{Z}) \xrightarrow{p} H_2(M, \partial M; \mathbb{Z}) \longrightarrow \dots.$$

Note that $H_2(M, \partial M; \mathbb{Z})$ is free abelian and hence its subgroups are free abelian as well. Therefore, we have a non-natural isomorphism:

$$H_2(M, A; \mathbb{Z}) \cong \operatorname{Im} p \oplus \ker p \cong \operatorname{Im} p \oplus \operatorname{Im} i.$$

Every element in $\operatorname{Im} p \subset H_2(M, \partial M; \mathbb{Z})$ and $\operatorname{Im} i \subset H_2(\partial M, A; \mathbb{Z})$ can be represented by an embedded surface in M. Now the theorem follows from the oriented sum construction (see Definition 1.3). $\square$

Thurston norm

Given a surface S with connected components $S_1, \dots S_n$, we define the complexity of the surface S by

$$\chi_-(S) = \sum_{i=1}^{n} \max\{0, -\chi(S_i)\}.$$

An alternative description of the complexity is given by removing all components of S which are discs and spheres and then take the negative Euler characteristic.

Let M be a compact oriented irreducible 3-manifold and $A \subset \partial M$ a subsurface. The Thurston norm is defined by

$$\begin{aligned} \|\cdot\|_{M,A} \colon H_2(M, A; \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \sigma &\longmapsto \min\left\{\chi_-(S) \;\middle|\; \begin{array}{l} [S] = \sigma \text{ and } S \text{ is properly} \\ \text{embedded i.e. } \partial S = S \cap A \end{array}\right\}. \end{aligned}$$

If A is all of the boundary i. e. $A = \partial M$, then we will drop the subscript and write $\|\cdot\|$ instead of $\|\cdot\|_{M,\partial M}$. In this case one can use Poincaré duality to define a function $\|\cdot\|$ on $H^1(M; \mathbb{Z})$.

Lemma 1.4. *Let M be a compact oriented irreducible 3-manifold with incompressible boundary. For all $n \in \mathbb{Z}$ and $\sigma_1,\ \sigma_2 \in H_2(M, \partial M; \mathbb{Z})$ one has*

1. *homogeneity:* $\|n \cdot \sigma_1\|_{M,\partial M} = |n| \cdot \|\sigma_1\|_{M,\partial M}$,

2. *subadditivity:* $\|\sigma_1 + \sigma_2\|_{M,\partial M} \leqslant \|\sigma_1\|_{M,\partial M} + \|\sigma_2\|_{M,\partial M}$.

Proof . (1) We suppose that σ_1 is primitive. Let S be surface in M with $[S] = \sigma_1$ and $\chi_-(S) = \|\sigma_1\|$. Since S is oriented, S has a trivial normal bundle. We can embed $|n|$ disjoint parallel copies of S using a trivialisation of the normal bundle. This gives the inequality $\|n \cdot \sigma_1\| \leqslant |n| \cdot \|\sigma_1\|$. For the converse inequality one has to recall the proof of Theorem 1.2. Given a surface Σ in M with $[\Sigma] = n \cdot \sigma_1$, one can construct a map $f\colon M \to S^1$ such that $\Sigma = f^{-1}(\{1\})$ and $D_M([\Sigma]) = D_M(n \cdot \sigma_1) = f^*(\theta)$, where D_M denotes the Poincaré duality isomorphism and θ denotes the standard generator of $H^1(S^1;\mathbb{Z})$. Note that S^1 is an Eilenberg-Maclane space. By the standard theory of Eilenberg-Maclane spaces and by construction one has that the image of the map $f_*\colon \pi_1(M) \to \pi_1(S^1) \cong \mathbb{Z}$ is equal to $n \cdot \mathbb{Z}$. By standard covering theory we obtain a lift $\overline{f}$ of f and a commutative diagram:

$$\begin{tikzcd} & S^1 \arrow[d, "z \mapsto z^n"] \\ M \arrow[ur, "\overline{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

Therefore, one has $\Sigma = f^{-1}(\{1\}) = \bigcup_{i=1}^{n} \overline{f}^{-1}(\{\zeta^i\})$, where ζ is a primitive n-th root of unity. Since each $f^{-1}(\{\zeta^i\})$ represents σ_1, we have proven the other inequality.
(2) Let S and T be proper submanifolds of M such that S represents σ_1 and T represents σ_2 and $\chi_-(S) = \|\sigma_1\|$ and $\chi_-(T) = \|\sigma_2\|$. Since M is irreducible and the boundary is incompressible, we can assume by the loop theorem that S and T does not contain sphere or disc components and that S and T are incompressible. After an isotopy we can assume that S and T are in general position. Suppose a circle of intersection is inessential in M, then it is inessential in T and S and hence bounds discs $D_1 \subset S$ and $D_2 \subset T$. By starting with an innermost circle, we can assume that $D_1 \cup D_2$ is an embedded sphere, which bounds a ball in M and whose interior is disjoint from S and T. By pushing one of the surfaces along the ball we can remove some intersection. Now if we assume that S and T are isotoped in a way that the number of components of $S \cap T$ is minimal, then $S \oplus T$ is an embedded surface without disc and sphere components. Hence $\|\sigma_1 + \sigma_2\| \leqslant \chi_-(S \oplus T) = \chi_-(S) + \chi_-(T) = \|\sigma_1\| + \|\sigma_2\|$. $\square$

We summarise some formal properties of the Thurston norm that are proven in Appendix A. Here formal means that we only use the fact that the norm takes integral values.

Proposition 1.5. *Let M be a compact oriented irreducible 3-manifold and $\|\cdot\|$ the Thurston norm on $H_2(M, \partial M; \mathbb{Z})$. One has that*

1. *the function $\|\cdot\|_{M,\partial M}$ uniquely extends to a semi-norm on $H_2(M, \partial M; \mathbb{R})$,*
2. *the unit norm ball $B := \{x \in H_2(M, \partial M; \mathbb{R}) \mid \|x\| \leqslant 1\}$ is a finite sided polyhedra,*
3. *$\|x + y\| = \|x\| + \|y\|$ if and only if $x/\|x\|$ and $y/\|y\|$ lie on a common face of the norm ball,*

4. *for every* $a, b \in H_2(M, \partial M; \mathbb{R})$ *there is a* $\lambda > 0$ *such that*
$$\|(\lambda x + y) + y\| = \|\lambda x + y\| + \|y\|.$$

Definition 1.6. Let M be a compact oriented irreducible 3-manifold and $A \subset \partial M$ a closed codimension 0 submanifold. We call a properly embedded surface S in M Thurston norm minimizing if S has no sphere or disc components, S is incompressible, all possible unions of components of S are homologically essential in $H_2(M, A; \mathbb{Z})$, and $\chi_-(S) = \|[S]\|_{M,A}$. If we do not specify A, then it is understood that $A = \partial M$.

Remark 1.7. The empty set $\emptyset$ is a Thurston norm minimizing representative of $0 \in H_2(M, \partial M; \mathbb{Z})$. Moreover, if M has incompressible boundary, then it is a consequence of the loop theorem that every class admits a Thurston norm minimizing representative [AFW15, Chapter 3 C.22].

Remark 1.8. It is a direct consequence of Lemma 1.4 that if the union $T \cup S$ of two surfaces T and S is Thurston norm minimizing, then S and T are Thurston norm minimizing.

For later references we also define a function using immersed surfaces:
$$\begin{aligned} \|\cdot\|_{\text{immersed}} \colon H_2(M, \partial M; \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \sigma &\longmapsto \min\left\{ \chi_-(S) \;\middle|\; \begin{array}{l} [S] = \sigma \text{ and } S \text{ is properly} \\ \text{immersed i.e. } \partial S = S \cap A \end{array} \right\}. \end{aligned}$$

One has the obvious inequality $\|\cdot\|_{\text{immersed}} \leqslant \|\cdot\|$, but we will see later that these two semi-norms coincide.

We end this section by showing that the Thurston norm behaves additive under gluing along tori.

Proposition 1.9. *Let* N *be a compact oriented irreducible 3-manifold with empty or toroidal boundary and let* Σ *be a Thurston norm minimizing surface. Let* $\mathcal{T} = T_1 \cup \ldots \cup T_n \to N$ *an embedding of incompressible tori in* N*. Let* $\nu \colon \mathcal{T} \times [-1, 1] \to N$ *be a tubular neighbourhood. Then there exists an isotopy of the embedding* $\Sigma \to N$ *such that for any connected component of* $M \subset N \setminus \nu(\mathcal{T} \times (-1, 1))$ *the intersection* $\Sigma \cap M$ *is a Thurston norm minimizing surface.*

Proof . Let Σ be Thurston norm minimizing. After an isotopy we can assume that the intersection of Σ with $\mathcal{T}$ is transversal and the number of connected components of the intersection is minimal. Since N is irreducible this implies that for each torus $T_i \subset \mathcal{T}$ every circle $C \subset \Sigma \cap T$ is essential. Let $M \subset N \setminus \nu(\mathcal{T} \times (-1, 1))$ be a connected component and denote by $S_M := \Sigma \cap M$ the properly embedded surface in M. Then by the discussion before S_M has no disc components. Since Σ and $\mathcal{T}$ intersect in as few components as possible, we see that S_M does not contain any boundary parallel annulus.

The rest of the proof is devoted to show that S_M has minimal complexity among the homology classes it represents. Therefore, we are more flexible than restricting ourselves to isotopies. Namely, we are allowed to replace Σ and hence S_M as long as we do not change the complexity or the homology class.

Now assume S' is a Thurston norm minimizing surface in M, which is homologous to S_M. Obviously one has $[\partial S'] = [\partial S_M] \in H_1(\partial M)$.

Claim. One can replace Σ without changing the complexity or homology class such that $\partial S' = \partial S_M$.

Note that ∂S_M are essential circles in ∂M and are equipped with an orientation coming from S_M. Two oriented essential circles on the torus which do not intersect are either parallel or are the boundary of an oriented annulus. If two circles of ∂S_M bound an embedded oriented annulus, then we can replace S_M by $S_M \cup A$ which is a homologous surface with the same complexity. We can push it off and call the resulting surface T. We do the same construction later a second time and refer to Figure 1.3 for an illustration.

We replace Σ by $(\Sigma \setminus S_M) \cup A \cup T$. The resulting surface is homologous and has the same complexity. After removing all such annuli, the equation $[\partial S'] = [\partial S_M] \in H_1(\partial M)$ implies that $\partial S'$ and ∂S_M have to be isotopic.

Now we assume that $\partial S' = \partial S_M$ and obtain a class $\Sigma' := \Sigma \setminus S_M \cup S'$ which is homologous to Σ. We have

$$\chi(\Sigma') \leqslant \chi(\Sigma) = \chi(\Sigma \setminus S_M) + \chi(S_M) \leqslant \chi(\Sigma \setminus S_M) + \chi(S') = \chi(\Sigma').$$

Since there are no disc components the complexity is equal to the negative Euler characteristic and hence S_M and S' have the same complexity. □

One can also prove a slightly stronger statement, but we only need it in a special case and hence we will only discuss this special case.

But first we recall a standard fact from 3-manifold topology, which is a direct consequence of the loop theorem.

Lemma 1.10. *[[AFW15, Justification C.22]] Let N be a connected compact oriented irreducible 3-manifold with non-empty toroidal boundary. If there is a base point $x \in \partial N$ such that the inclusion induced map $\pi_1(\partial N, x) \to \pi_1(N, x)$ is not injective, then $N \cong S^1 \times D^2$.*

Lemma 1.11. *Let $N \neq S^1 \times D^2$ be a compact oriented irreducible 3-manifold with non-empty toroidal boundary. Denote by $D(N)$ the double of N i. e.*

$$D(N) := N_1 \bigsqcup_{\partial N_1 = \partial N_2} N_2$$

with $N_i := N \times \{i\}$. If Σ is Thurston norm minimizing in N, then the double $D(\Sigma)$ is Thurston norm minimizing in $D(N)$. (Here $D(\Sigma) = \Sigma_1 \cup \Sigma_2$, where Σ_i is the image of Σ in N_i.)

Proof of Lemma 1.11 . Let S be a Thurston norm minimizing surface in $D(N)$ with $[D(\Sigma)] = [S] \in H_2(D(N);\mathbb{Z})$. We write $S_i \coloneqq S \cap N_i$. By Lemma 1.10 we can assume that ∂N_i is incompressible in N_i and by Proposition 1.9 we can assume that S is in general position such that S_i is Thurston norm minimizing in N_i. Note that $2\cdot\chi(\Sigma) = \chi(D(\Sigma))$ and $\chi(S_1) + \chi(S_2) = \chi(S)$. We are left to prove $[\Sigma_i] = [S_i] \in H_2(N_i, \partial N_i;\mathbb{Z})$. Then Σ_i and S_i are Thurston norm minimizing for the same class and we would get $\chi(S_i) = \chi(\Sigma)$. The equation $[\Sigma_i] = [S_i]$ follows from a slightly more general statement.

Claim. Let S be an embedded surface in $D(N)$, which is in general position with $\partial N \subset D(N)$. Denote by $p\colon\ H_2(D(N);\mathbb{Z}) \to H_2(D(N), N_2;\mathbb{Z})$ the projection map and $ex\colon\ H_2(D(N), N_2;\mathbb{Z}) \to H_2(N_1, \partial N_1;\mathbb{Z})$ the inverse of the excision isomorphism, then $ex \circ p([S]) = [S \cap N_1]$.

The idea to prove the claim is to approximate $[S]$ by simplexes, which lie completely either in N_1 or in N_2. Then the statement follows from the proof of the excision theorem. To make this precise one needs to work with tubular neighbourhoods. We leave the nasty technical detail to the ambitious reader. □

Geometric properties of the Thurston norm

In this thesis we characterise embedded surface which are Thurston norm minimizing. One of the first criteria is due to Thurston, who shows that the compact leave of a taut foliation is Thurston norm minimizing. We refer to Calegari's book [Ca07] for an introduction to foliations in the context of 3-manifolds.

In his foundational paper on the Thurston norm Thurston proves the following theorem.

Theorem 1.12 ([Th86, Corollary 1])**.** *Let $\mathcal{F}$ be a taut oriented foliation on a closed oriented irreducible 3-manifold M, and S an immersed oriented surface. Then the following inequality holds*

$$|e(T\mathcal{F}) \frown [S]| \leqslant |\chi(S)|$$

with equality if and only if S is either homotopic into a leaf of $\mathcal{F}$ or is homotopic to a surface, all of whose tangencies with $\mathcal{F}$ are saddle tangencies with the same sign.

Remark 1.13. The left-hand side of the inequality only depends on homological data and since M is assumed to be irreducible we can get rid of all spherical components. Therefore, one obtains the inequality $|e(T\mathcal{F}) \frown [S]| \leqslant \|[S]\|_{\text{immersed}}$. This shows that if S is a leaf of a taut foliation that no component is a sphere, then S is norm minimizing with respect to immersed surfaces.

Sketch of the argument for Theorem 1.12 . We will restrict to the case that the foliation is smooth. The statement seems to be true in a general context of continuous Reebless foliations and 3-manifolds with boundary but it is hard to find a good reference.

The basic idea of the proof is the following. First one has to prove a general position theorem, which states that S is homotopic to an immersion and the only singular intersections of S and $\mathcal{F}$ are saddle tangencies. Then we can define the numbers:

$$\begin{aligned} I_p(S) &:= \text{number of positive saddle singularities} \\ I_n(S) &:= \text{number of negative saddle singularities} \end{aligned}$$

In this case the foliation of M pulls back to a foliation of S with isolated singularities. This can be interpreted as a vector field on S with isolated zeros. Since $\mathcal{F}$ is transversally oriented, all the zeros will have the same index. By the Poincaré–Hopf formula we get $|I_p(S) + I_n(S)| = |\chi(S)|$. By the geometric interpretation of the cap-product one has $e(T\mathcal{F}) \frown [S] = I_p(S) - I_n(S)$. $\square$

In the next section we discuss a sort of converse of Theorem 1.12.

1.2. Sutured manifolds

If M is an oriented manifold, then we endow ∂M with the orientation coming from the outwards-pointing normal vectors first. A sutured manifold (M, R_+, R_-, γ) is a compact oriented 3-manifold with a decomposition of its boundary

$$\partial M = R_+ \cup \gamma \cup -R_-,$$

into oriented submanifolds such that

1. γ is a collection of disjoint embedded annuli or tori,
2. $R_+ \cap R_- = \emptyset$,
3. if A is an annulus component of γ, then $R_- \cap A$ is a boundary component of A and of R_-, and similarly for $R_+ \cap A$. Furthermore, $[R_+ \cap A] = [R_- \cap A] \in H_1(A; \mathbb{Z})$ where we endow $R_\pm \cap A$ with the orientation coming from the boundary of the oriented manifold $R_\pm$.

Remark 1.14. Taking the orientation into account one has $[R_+] = [R_-] \in H_2(M, \gamma; \mathbb{Z})$.

We call a sutured manifold M taut, if M is irreducible and R_+ and R_- are Thurston norm minimizing viewed as properly embedded surfaces in (M, γ). We call a sutured manifold balanced if $\chi_-(R_+) = \chi_-(R_-)$.

Remark 1.15. In our definition of a taut sutured manifold we demand $R_\pm$ to be incompressible, which differs from the convention most other authors choose. Our convention just rules out notorious counterexamples in the case that $M = S^1 \times D^2$.

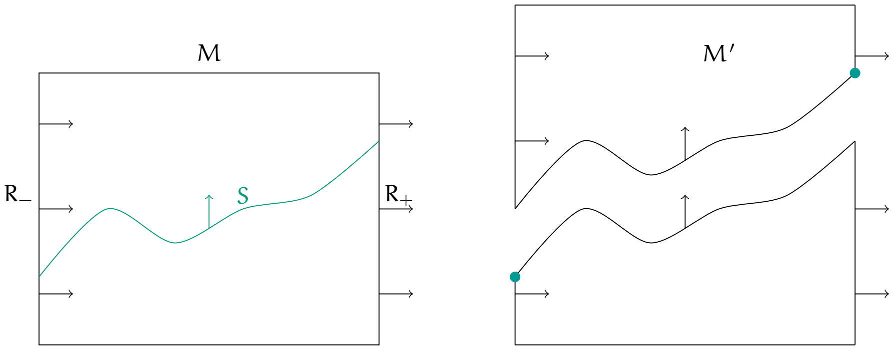

Figure 1.2.: This picture shows a sutured manifold decomposition. The manifold M is cut along S. The dark green dots indicate where the orientation of $R_\pm$ and S do not match. Hence they correspond to sutures.

An example of a balanced taut sutured manifold is given by the *product sutured manifold*

$$(R \times [-1, 1], R \times \{1\}, R \times \{-1\}, \partial R \times [-1, 1]),$$

where R is a surface with $R \ncong S^2$.

Gabai [Ga83] introduced the notation of a sutured manifold decomposition, which we recall now. Let $(S, \partial S)$ be a properly embedded incompressible oriented surface in a sutured manifold (M, R_+, R_-, γ). We call S a decomposition surface if S is transverse to $R_\pm$ and for every connected component $c \in S \cap \gamma$ one of the following holds.

1. c is a properly embedded non-separating arc,
2. c is a simple closed curve in an annulus component of γ which is homologous to $[R_- \cap A] \in H_1(A; \mathbb{Z})$,
3. c is a homotopically non-trivial curve in a torus component T of γ and if c' is another curve in $S \cap T$, then c and c' are homologous in T.

Given a decomposition surface S, we define the sutured decomposition along S by

$$(M, R_-, R_+, \gamma) \overset{S}{\rightsquigarrow} (M', R'_-, R'_+, \gamma')$$

where

$$\begin{aligned}
M' &= M \setminus S \times (-1, 1), \\
\gamma' &= (\gamma \cap M') \cup \overline{\nu(S'_+ \cap R_-)} \cup \overline{\nu(S'_- \cap R_+)}, \\
R'_+ &= ((R_+ \cap M') \cup S'_+) \setminus \operatorname{int} \gamma', \\
R'_- &= ((R_- \cap M') \cup S'_-) \setminus \operatorname{int} \gamma'.
\end{aligned}$$

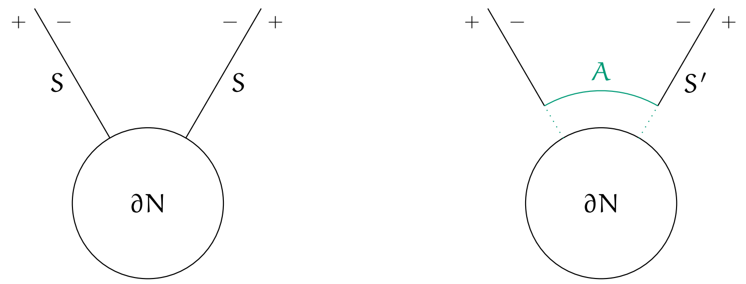

Figure 1.3.: A Thurston norm minimizing surface can be turned into a decomposition surface.

Here S'_+ (resp. S'_-) is the outward-pointing (resp. inward-pointing) part of $S \times \{-1,1\} \cap M'$ (see Figure 1.2). We sometimes abbreviate the notation of (M, R_+, R_-, γ) to (M, γ) and refer to R_+ and R_- by $R_+(\gamma)$ and $R_-(\gamma)$. Moreover, $s(\gamma)$ denotes for each annulus component of γ an essential simple closed curve matching the orientation of γ.

We make use of the following elementary lemma rather frequently. The proof is left as an exercise.

Lemma 1.16. *Let $(N, \emptyset, \emptyset, \partial N)$ be a taut sutured manifold (i. e. N is compact oriented irreducible 3-manifold with empty or toroidal boundary) and let S be a Thurston norm minimizing surface in N such that S is a decomposition surface. Then N' defined by $N \overset{S}{\rightsquigarrow} N'$ is a taut sutured manifold.*

If N is closed, then obviously every Thurston norm minimizing surface is a decomposition surface. If N has non-empty boundary, then a Thurston norm minimizing surface may violate Condition (3) in the definition of decomposition surface. We hence introduce the following definition.

Definition 1.17 (Taut surface)**.** Let N be a compact oriented irreducible 3-manifold with empty or toroidal boundary. A Thurston norm minimizing surface S is called taut if S is a decomposition surface for the sutured manifold $(N, \emptyset, \emptyset, \partial N)$.

Remark 1.18. Usually the distinction between taut and Thurston norm minimizing does not matter much. However, a version of Proposition 1.9 for taut surfaces does not hold in general.

Lemma 1.19. *If $N \neq S^1 \times D^2$ is a compact oriented irreducible 3-manifold with empty or toroidal boundary, then every class $\sigma \in H_2(N, \partial N; \mathbb{Z})$ can be represented by a taut surface.*

Proof . We already discussed in Remark 1.7 that every class can be represented by a Thurston norm minimizing surface. Therefore, we take a Thurston norm minimizing surface S with $\pi_0(\partial S)$ minimal among all surfaces representing the class σ. In this case S is taut as can be seen as follows. Suppose the contrary i. e. S is not a decomposition surface. Then there are two components of ∂S bounding an oriented annulus A in ∂N. We can define the surface $S' := S \cup A$ which has the same complexity as S and which represents the same homology class. One can isotope S' to make it properly embedded (see Figure 1.3). Moreover, one has $\pi_0(\partial S') = \pi_0(\partial S) - 2$ which contradicts our choice of S. $\square$

We also need the following lemma from the theory of sutured manifold decomposition due to Gabai.

Lemma 1.20. *Let N be connected compact oriented irreducible 3-manifold with empty or toroidal boundary. Let S and F be taut surfaces in general position such that the number of components of $S \cap F$ is minimal. We assume that $S \oplus F$ is taut.*

Denote by N' the sutured manifold obtained by $N \overset{S}{\rightsquigarrow} N'$, then $F' := F \cap N'$ is a decomposition surface for N'. Moreover, N' and N'' are taut sutured manifolds, where N'' is given by $N' \overset{F'}{\rightsquigarrow} N''$. One also has a commutative diagram of taut sutured manifold decompositions

$$\begin{tikzcd} N \arrow[r, rightsquigarrow, "S"] \arrow[d, rightsquigarrow, "F \oplus S"] & N' \arrow[d, rightsquigarrow, "F'"] \\ N \backslash\backslash (F \oplus S) \arrow[r, rightsquigarrow, "C"] & N'' \end{tikzcd}$$

where $F \oplus S$ is the oriented sum (see Definition 1.3) and $C = C_1, \dots C_n$ is a disjoint union of annuli and discs.

The proof will mostly consists of the following lemma.

Lemma 1.21. *[Gabai Lemma 3.12] Let $(M, \gamma) \overset{J}{\rightsquigarrow} (M', \gamma')$ be a decomposition surface such that either J is a disc and $|J \cap s(\gamma)| = 2$ or J is an annulus with one component of ∂J lying in each of R_+ and R_-. Then (M, γ) is taut if and only if (M', γ') is taut.*

Proof . We sketch the case that J is a disc. The assumption that $|J \cap s(\gamma)| = 2$ ensures that ∂J intersects R_+ (resp. R_-) in exactly one properly embedded arc. There might be two cases. In the first case J intersects the same suture twice or J intersects two sutured exactly once. In both cases one has $\|R_\pm(\gamma')\| + 1 = \|R_\pm(\gamma)\|$, because in both cases one removes a disc D. But the resulting sutures γ' are different according to the cases. Let T be a surface properly embedded in (M', γ'), homologous to $R_+(\gamma')$, and $\chi_-(T) \leqslant \chi_-(R_+(\gamma'))$. Since T is homologous to $R_+(\gamma')$ we can assume that $\partial T = s(\gamma')$. We construct a surface S by taking T and glue in the disc D removed before. Then S is homologous to $R_+(\gamma)$ and $\chi_-(S) = \chi_-(T) + 1 \leqslant \|R_+(\gamma)\|$. $\square$

Proof of Lemma 1.20 . The fact that F' is a decomposition surface follows from the assumption that $S \oplus F$ is a decomposition surface. The union C of discs and annuli are corresponding to the components of $F \cap S$. Thus this lemma follows from Lemma 1.16, Figure 1.4, and Lemma 1.21. $\square$

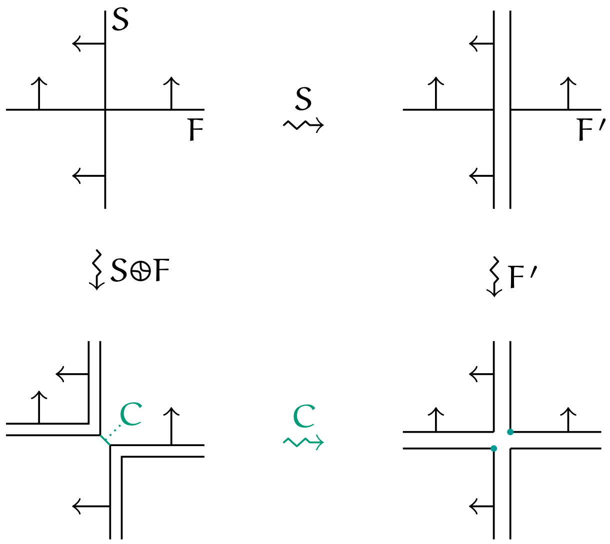

Figure 1.4.: We illustrate the commutativity of the sutured manifold decomposition. The collection C of annuli and discs is in correspondence with $S \cap F$.

The work of Gabai

Gabai proved a powerful equivalence between the combinatorial data of a sutured manifold and differential geometric data of a taut foliation. In order to state the equivalence, we need one more definition.

Definition 1.22. [Taut foliations on sutured manifolds] A foliation $\mathcal{F}$ of a sutured manifold (M, R_+, R_-, γ) is called taut if

1. $\mathcal{F}$ is co-oriented and components of R_+ and R_- are leaves, whose transverse orientation agrees with the label $R_\pm$.

2. γ is the subset of ∂M which is transverse to $\mathcal{F}$. Moreover, the foliation on each component A of γ induced by $\mathcal{F}$ is transverse to a foliation of S induced by a fibration over S^1.

3. Each leaf of $\mathcal{F}$ meets either a closed transverse circle or a compact, properly embedded transverse arc with one endpoint in R_+ and the other in R_-.

Remark 1.23. If we consider a closed 3-manifold N as a sutured manifold, then one usually demand the existence of one transverse loop intersecting each leaf. In the definition above we only demand one transverse loop for every leaf separately. We refer to the book of Calegari [Ca07, Lemma 4.26] for a proof that these definitions are equivalent.

The fundamental result of Gabai is the following equivalence.

Theorem 1.24 ([Ga83, Corollary 5.3]). *Let (M, γ) be a sutured manifold. If M is not equal to B^3 or $S^1 \times S^2$ and $H_2(M, \gamma) \neq 0$, then the following are equivalent:*

1. *(M, γ) is taut,*
2. *M admits a taut foliation tangent to $R_\pm(\gamma)$ and transverse to γ which is smooth except for neighbourhoods of torus components in $R_\pm(\gamma)$.*

Idea of the proof . The proof is an induction argument. Gabai first shows the statement for a product sutured manifold which is the induction beginning. Then he considers a sutured manifold (M, R_-, R_+, γ) and a decomposition surface S. The next thing he shows is that if the theorem holds for $(M', R'_-, R'_+, \gamma')$, where M' is obtained by $(M, R_-, R_+, \gamma) \stackrel{S}{\rightsquigarrow} (M', R'_-, R'_+, \gamma')$, then the theorem holds for M. This is the induction step.

The difficulty is to show that for every taut sutured manifold one finds a sequence of sutured manifold decompositions which end in product sutured manifolds. We refer to Gabai's original work [Ga83] for the details. □

This theorem has some very nice corollaries which we now collect.

Corollary 1.25. *Let M be a compact oriented irreducible 3-manifold with empty or toroidal boundary and S a taut surface. Then M admits a taut foliation $\mathcal{F}$ such that S is a compact leave. The foliation is smooth except for toroidal components of S.*

Proof . If we consider the sutured manifold decomposition $M \stackrel{S}{\rightsquigarrow} M'$, then M' is a taut sutured manifold. □

If $p\colon M \to N$ is a finite cover, then the pull back of a taut foliation is again a taut foliation in the sense of Definition 1.22. So we obtain the next corollary.

Corollary 1.26. *If (N, γ) is a taut sutured manifold and $p\colon M \to N$ a finite cover, then $(M, p^{-1}(\gamma))$ is a taut sutured manifold.*

Another statement which follows from the fact that the pull back of a taut foliation is taut is the next corollary.

Corollary 1.27. *Let N be a compact oriented irreducible 3-manifold and $p\colon M \to N$ be a finite cover. Let S be a properly embedded surface in N. If S is taut, then $p^{-1}(S)$ is taut.*

We can also show that the Thurston norm and the norm using immersed surfaces agree, since Theorem 1.12 holds for immersed surfaces.

Corollary 1.28. *Let M be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary. We have $\| \cdot \| = \| \cdot \|_{immersed}$.*

Proof . We first assume that M is closed and S is a taut surface that has no torus component. Then the statement follows from Corollary 1.25 and Theorem 1.12.

Now adding a torus components to a homology class neither changes the immersed norm nor the Thurston norm. If M has toroidal boundary, then one can use the doubling argument of Lemma 1.11. □

One can also compare the Thurston norm to the L^1-norm or sometimes called Gromov norm. We recall the definition here. Let X be a topological space and $c \in C_k(X;\mathbb{R})$ a chain. Then c is a sum of simplexes $c = \sum_i^n n_i \cdot \sigma_i$ to which we assign the value $\|c\|_1 \coloneqq \sum_{i=1}^n |n_i|$.

The L^1-(semi)-norm for an element $\sigma \in H_k(X;\mathbb{Z})$ is defined by

$$\|\sigma\|_1 = \inf\{\|c\|_1 \mid c \text{ cycle in } C_k(X;\mathbb{R}) \text{ and } [c] = \sigma\}.$$

By a result of Gromov one has for the fundamental class $[S] \in H_2(S;\mathbb{Z})$ of a surface S with negative Euler-characteristic that

$$\|[S]\|_1 = 2 \cdot |\chi(S)|.$$

From Corollary 1.28 together with Lemma 1.1 one has

Corollary 1.29. *If M is a closed oriented irreducible 3-manifold, then $\|\cdot\|_1 = 2\|\cdot\|$.*

Proof . By Lemma A.6 it is sufficient to prove the statement only for integral classes. Given $\sigma \in H_2(M;\mathbb{Z})$, we get for every $\epsilon > 0$ a cycle $c_\epsilon \in C_2(M;\mathbb{R})$ representing σ such that $\big|\|\sigma\|_1 - \|c_\epsilon\|_1\big| \leqslant \epsilon$. We can actually assume that $c_\epsilon \in C_2(M;\mathbb{Q})$. Now there is $k_\epsilon \in \mathbb{N}$ such that $k_\epsilon \cdot c_\epsilon \in C_2(M;\mathbb{Z})$. Using $k_\epsilon \cdot c_\epsilon$ and Lemma 1.1 we can construct a map $f\colon \Sigma_\epsilon \to M$ with $f_*([\Sigma_\epsilon]) = k_\epsilon \cdot \sigma$ and $-\chi(\Sigma) \leqslant \frac{1}{2}\|k_\epsilon \cdot c_\epsilon\|_1$. Since Σ_ϵ and M are smooth manifolds f is homotopic to a smooth immersion g. Therefore, we have

$$\|\sigma\|_{\text{immersed}} \leqslant \frac{\chi_-(\Sigma)}{k_\epsilon} \leqslant \frac{1}{2}\|c_\epsilon\|_1 \leqslant \frac{1}{2}\|\sigma\|_1 + \epsilon.$$

This inequality holds for all $\epsilon > 0$ and hence $\|\sigma\|_{\text{immersed}} \leqslant \frac{1}{2}\|\sigma\|_1$. Conversely, if we have an immersion $g\colon \Sigma \to M$ with $g_*([\Sigma]) = \sigma$ and $-\chi(\Sigma) = \|\sigma\|_{\text{immersed}}$, then we can approximate the fundamental class $[\Sigma]$ by a sequence of cycles $c_n \in C_2(\Sigma;\mathbb{R})$ representing $[\Sigma]$ and such that $\lim_{n\to\infty}\|c_n\|_1 = -2\cdot\chi(\Sigma)$. We conclude

$$\|\sigma\|_1 \leqslant \lim_{n\to\infty}\|g_*(c_n)\|_1 = \lim_{n\to\infty}\|c_n\|_1 = -2\cdot\chi(\Sigma) = 2\cdot\|\sigma\|_{\text{immersed}}. \qquad \square$$

1.3. Cut along a hypersurface and cyclic covers

In the section we introduce the notation of „cut along" a hypersurface. This is a frequently used operation in low-dimensional topology. However, our treatment here is for any dimension.

Let W be an oriented smooth n-manifold and $i\colon X \to W$ be a proper embedding of an oriented submanifold of codimension 1. We call such an X a hypersurface. Since X is oriented, it comes with a trivialisation of the normal bundle. We obtain an embedding $\nu\colon X \times [-1,1] \to W$. We consider the topological space $W \setminus \nu(X \times (-1,1))$. This space comes with two canonical embeddings $i_\pm\colon X \to W \setminus \nu(X \times (-1,1))$ given by the map $X \to X \times \{1\}$ (resp. $X \to X \times \{-1\}$) composed with ν. We denote the resulting images by X_+ and X_-. We obtain the space W back from $W \setminus \nu(X \times (-1,1))$ by gluing X_+ to X_-. In particular, there is a map $\psi\colon W \setminus \nu(X \times (-1,1)) \to W$ such that the diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "i_+ \sqcup i_-"] \arrow[d] & W \setminus \nu(X \times (-1,1)) \arrow[d, "\psi"] \\ X \arrow[r, "i"] & W \end{tikzcd}$$

is a push-out diagram.

Definition 1.30. By the terminology W is cut along X is meant the following data:

1. the topological space $W \setminus \nu(X \times (-1,1))$,
2. the inclusion maps $i_\pm$,
3. and the projection map $\psi\colon W \setminus \nu(X \times (-1,1)) \to W$.

In symbols we write $W \setminus\setminus X$.

From now on we are a bit sloppy about notation and mostly view $W \setminus\setminus X$ as the space $W \setminus \nu(X \times (-1,1))$. But we will illustrate that it is useful to remember more data.

Lemma 1.31. *Let W be an oriented connected smooth n-manifold and X a hypersurface. Let $x_0 \in W$ be a base point disjoint from X. Let $\phi \in H^1(W;\mathbb{Z})$ be the Poincaré dual of $i_*([X]) \in H_{n-1}(W, \partial W;\mathbb{Z})$. By abuse of notation, we write $\phi\colon \pi_1(W) \to \mathbb{Z}$ for the group homomorphism given by ϕ under the chain of isomorphisms*

$$H^1(W;\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(W),\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(W,x_0),\mathbb{Z}).$$

We consider the space

$$W_\phi := W \setminus\setminus X \times \mathbb{Z} / \sim_R,$$

where the equivalence relation is given by

$$\begin{aligned} (x,j) \sim_R (y,k) \quad :\Leftrightarrow \quad & (x,j) = (y,k) \textit{ or} \\ & \exists z \in N : i_+(z) = x,\ i_-(z) = y,\ k = j+1. \end{aligned}$$

If $\psi\colon W \setminus\setminus X \to W$ denotes the projection map, then the map

$$\begin{aligned} q\colon W_\phi &\longrightarrow W \\ (x,i) &\longmapsto \psi(x) \end{aligned}$$

is well-defined and makes W_ϕ a cover of W, which is isomorphic to $\mathbb{Z} \times_\phi \widetilde{W}$ as covering spaces.

We refer to Figure 1.5 for an example of this construction.

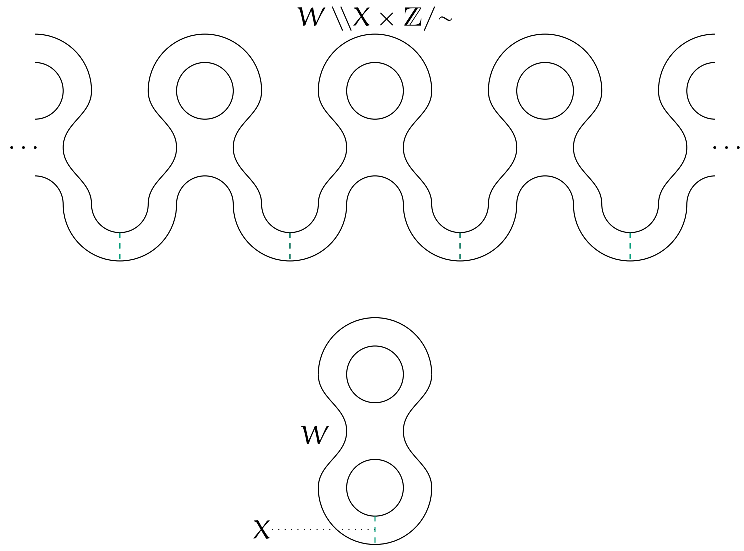

Figure 1.5.: An example of the construction described in Lemma 1.31.

Sketch of the argument . We only describe the maps between W_ϕ and $\mathbb{Z} \times_\phi \widetilde{W}$. To check that the maps are continuous is left as a cumbersome exercise. Note that by general position every path γ in W is homotopic to a path transverse to X i. e. $\gamma(I)$ intersects X transversely in finitely many points. Denote by $\#(\gamma \cap X) \in \mathbb{Z}$ the algebraic intersection number. For every $x \in W \setminus\setminus X$ let $\gamma_x \colon I \to W$ be a smooth path from x_0 to $\psi(x)$ which is transverse to X. We define a map

$$\begin{aligned} \Theta\colon W_\phi &\longrightarrow \mathbb{Z} \times_\phi \widetilde{W} \\ (x, i) &\longmapsto \begin{cases} (i - \#(\gamma_x \cap X), \gamma_x) & x \notin X_- \\ ((i-1) - \#(\gamma_x \cap X), \gamma_x) & x \in X_- \end{cases}. \end{aligned}$$

Basically by definition this map is independent of the choice of representative of (x, i). What takes a bit more effort is the fact that the map is also independent of the chosen path.

Claim. The definition of Θ is independent of the choice of γ_x.

Let η_x be another path from x_0 to $\psi(x)$ which is in general position to X. If $\overline{\eta_x}$ denotes the inverse path of η_x, then $\gamma_x * \overline{\eta_x}$ is an element in $\pi_1(W, x_0)$. It is well-known that [Br93, Theorem 11.9] :

$$\phi(\gamma_x * \overline{\eta_x}) = \#(\gamma_x \cap X) + \#(\overline{\eta_x} \cap X) = \#(\gamma_x \cap X) - \#(\eta_x \cap X). \tag{1.1}$$

Hence we have the chain of equalities

$$\begin{aligned}\Theta(x,i) &= (i - \#(\gamma_x \cap X), \gamma_x) \\ &= (i - \#(\gamma_x \cap X), \gamma_x * \overline{\eta_x} * \eta_x) \\ &= (i - \#(\gamma_x \cap X), (\gamma_x * \overline{\eta_x}) \cdot \eta_x) \\ &= (i - \#(\gamma_x \cap X + \phi(\gamma_x * \overline{\eta_x})), \eta_x) \\ &= (i - \#(\eta_x \cap X), \eta_x)\,.\end{aligned}$$

The second equality comes from the fact that γ_x and $\gamma_x * \overline{\eta_x} * \eta_x$ are homotopic and hence correspond to the same point in $\widetilde{X}$. The third equality is the definition of the action of $\pi_1(W, x_0)$ on $\widetilde{W}$, the fourth equality comes from the equivalence relation on $\mathbb{Z} \times_\phi \widetilde{X}$, and the last equality comes from Equation (1.1). This proves the claim.

Next we construct an inverse map. If we have a path γ in W, then we will assume it is in general position to X. Moreover, we have that $\psi\colon (W \setminus\!\!\setminus X) \setminus X_- \to W$ is a bijection and we denote by $\psi^{-1}\colon W \to (W \setminus\!\!\setminus X) \setminus X_-$ the inverse (which is in general not continuous at points in X). Now we can construct the inverse of Θ by

$$\begin{aligned}\Xi\colon \mathbb{Z} \times_\phi \widetilde{W} &\longrightarrow W_\phi \\ (n, \gamma) &\longmapsto \big(n + \#(\gamma \cap X), \psi^{-1}(\gamma(1))\big)\,.\end{aligned}$$

This is a well-defined map, since for any $g \in \pi_1(W, x_0)$ we have $\#(g * \gamma \cap X) = \phi(g) + \#(\gamma \cap X)$. It is easy to check that Θ and Ξ are inverse to each other, but one still has to check that both maps are continuous. As mentioned in the beginning this is left as an exercise. □

A graph associated to a hypersurface

Next we describe a graph to keep track of the combinatorics of $W \setminus\!\!\setminus X$.

Definition 1.32. Let W be a smooth oriented n-dimensional manifold and X be a (not necessarily connected) hypersurface. Recall that by definition a hypersurface is oriented. We obtain a directed multigraph $G(W, X)$ in the following way. The set of vertices V is given by the connected components of $W \setminus\!\!\setminus X$. The edges correspond to the connected components of X. Let M_1 and M_2 be connected components of $W \setminus\!\!\setminus X$. We put an edge from M_1 to M_2 for any connected component X' of X with $i_+(X') \subset M_1$ and $i_-(X') \subset M_2$. See Figure 1.6 for an example.

If W is connected and for any choice of base point $x \in X$ the inclusion induced map $\pi_1(X, x) \to \pi_1(W, x)$ is a monomorphism, then we obtain a presentation of the fundamental group of W as a graph of groups by assigning to each vertex M_j of $G(W, X)$ the group $\pi_1(M_j)$ and to each edge $X_i \subset X$ the group $\pi_1(X_i)$ together with the group homomorphisms induced from the inclusions $i_\pm\colon X_i \to M_j$. We do not need the precise notation of graph of groups. We only need the following theorem. A proof can be found in Serre's book [Se80, Chapter 1 Section 5.2 Corollary 1].

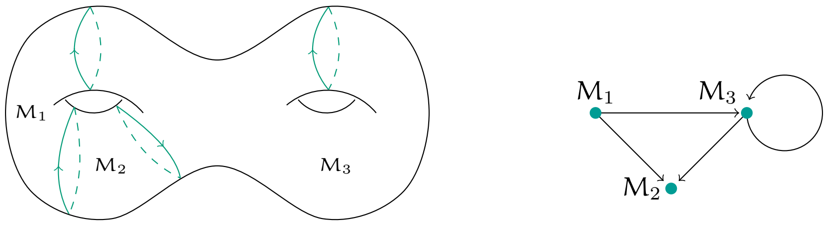

Figure 1.6.: The union of the green curves build the manifold X in W. The graph on the right is the associated graph $G(W, X)$.

Theorem 1.33. *If for any choice of base point $x \in X$ the map $\pi_1(X, x) \to \pi_1(W, x)$ is a monomorphism, then for each connected component $M \subset W \setminus\setminus X$ the map $\pi_1(M) \to \pi_1(W)$ is a monomorphism.*

Weighted hypersurface and its associated graph

A weighted hypersurface $\widehat{X} = (X_i, w_i)_{i\in\{1,\dots k\}}$ in a smooth oriented manifold W is a collection of pairs (X_i, w_i), where the X_i's are disjoint connected oriented hypersurfaces in W and w_i are positive integers.

Every weighted hypersurface $\widehat{X}$ defines a homology class $[\widehat{X}] := \sum_{i=1}^{n} w_i \cdot [X_i] \in H_2(W, \partial W; \mathbb{Z})$. By taking w_i parallel copies of X_i we get a properly embedded surface X such that $[X] = [\widehat{X}]$.

Note that every properly embedded hypersurface X in a manifold W can be seen as a weighted hypersurface by giving every component the weight 1.

Given a weighted surface $(X_i, w_i)_{i\in\{1,\dots k\}}$, we obtain a weighted directed graph by assigning to each edge of the graph $G(W, \bigcup_{i=1}^{k} X_i)$ (see Definition 1.32) the weight w_i.

We recall here a lemma of Turaev [Tu02, Lemma 1.2] which turns every hypersurface X in W in a weighted surface, where one removes all unnecessary connected components of X.

Proposition 1.34. *If $(X_i, w_i)_{i\in\{1,\dots k\}}$ is a weighted hypersurface in a smooth connected oriented n-dimensional manifold W, then there exists a subset $E \subset \{1, \dots k\}$ and weights $\{w'_e\}_{e\in E}$ such that the weighted surface $(X_e, w'_e)_{e\in E}$ has the following properties:*

1. $\sum_{i=1}^{k} w_i \cdot [X_i] = \sum_{e\in E} w'_e \cdot [X_e] \in H_2(W, \partial W; \mathbb{Z})$,
2. $W \setminus\setminus \bigcup_{e\in E} X_e$ *is connected.*

Proof . Given a weighted hypersurface $(X_i, w_i)_{i\in\{1,\dots k\}}$ in W, we write $X' = \bigcup_{i=1}^{k} X_i$ and consider the weighted graph $G(W, X')$.

Recall that edges are in one-to-one correspondence with the connected components $X_1, \dots, X_k$ of X and the vertices are in one-to-one correspondence to the connected components of $W \setminus\setminus X$. Hence the second bullet point of the statement is equivalent to the fact, that the graph $G(W, X')$ has only one vertex. So we prove this proposition by induction on the number of vertices of $G(W, X')$.

We write $X_e \subset X$ for the connected component corresponding to the edge e and we write $M_v \subset W \setminus\setminus X$ for the connected component corresponding to the vertex v.

In homology we have the equation

$$[\partial M_{v_1}] = \sum_{e=(v_1,v)} [X_e] - \sum_{e'=(v,v_1)} [X_{e'}] = 0 \in H_{n-1}(W, \partial W; \mathbb{Z}). \tag{1.2}$$

Now let $(X_i, w_i)_{i\in\{1,\dots k\}}$ be a weighted surface such that $G(W, X')$ has more than one vertex.

Since W is connected the graph $G(W, X')$ is connected. Let e_0 be an edge with minimal weight w_0 joining two distinct vertices v_1 and v_2 . Without loss of generality we assume that e_0 goes from v_1 to v_2. We define new weights

$$w'_e = \begin{cases} w_e + w_{e_0} & e = (v, v_1) \text{ and } v \neq v_1 \\ w_e - w_{e_0} & e = (v_1, v) \text{ and } v \neq v_1 \\ w_e & \text{else.} \end{cases}$$

We take E to be the subset of $\{1, \dots, k\}$ such that

$$e \in E \quad \Leftrightarrow \quad w'_e \neq 0.$$

By Equation 1.2 we have

$$[X] = \sum_{i=1}^{n} w_i [X_i] = \sum_{e\in E} w'_e [X_e].$$

Moreover, the graph $G(W, \bigcup_{e\in E} X_e)$ has at least one vertex less than $G(W, X')$, because M_{v_1} and M_{v_2} are connected in $W \setminus\setminus \bigcup_{e\in E} X_e$. This proves the induction step and hence this proposition. □

Remark 1.35. As mentioned earlier we can view every hypersurface X as a weighted surface, where every connected component has weight 1. Therefore, if X is a taut surface in a 3-manifold N, then the surface $\bigcup_{e\in E} X_e$ is taut as well (see Remark 1.8).

1.4. Virtual fibring and quasi-fibres

A lot of the material in this section can already be found in the preprint of the author [He18]. Here we introduce the notation of a quasi-fibre which is a special Thurston norm minimizing surface. We also study taut sutured manifolds, which are obtained by a sutured decomposition along a quasi-fibre. The main result of this section is Proposition 1.42.

Definition 1.36 (Quasi-fibre)**.** A taut surface Σ in a connected compact oriented irreducible 3-manifold is called quasi-fibre if there is a fibration $p\colon N \to S^1$ with fibre $F \subset N$ such that the surface $\Sigma \oplus F$ is taut.

Waldhausen studied incompressible surface in product 3-manifolds. We recall some of his results that are useful in the study of quasi-fibres. Let F be a connected surface possibly with boundary and $F \ncong S^2$. We endow $F \times [-1,1]$ with the product sutured manifold structure $(F, F_+, F_-, \partial F \times I)$, where $F_+ = F \times \{1\}$ and $F_- = F \times \{-1\}$. We denote by $p\colon F \times [-1,1] \to F$ the canonical projection. A properly embedded surface S in $F \times [-1,1]$ is called horizontal if $p|_S$ is a homeomorphism onto its image.

Remark 1.37. A horizontal surface S is by definition homeomorphic to a subsurface of F, so that we can view it as an embedding $S \to F \times [-1,1], x \mapsto (x, h(x))$. Therefore, S is isotopic to a subsurface of F_- (resp. F_+) by pushing (resp. lifting) the interval factor.

Given a connected incompressible decomposition surface S in $F \times [-1,1]$, there are evidently two possibilities:

1. S intersects F_+ and F_-,

2. $S \cap F_+ = \emptyset$ or $S \cap F_- = \emptyset$.

Waldhausen [Wa68, Proposition 3.1] showed that in the second case if $S \cap F_+ = \emptyset$ (resp. $S \cap F_- = \emptyset$), then S is ambient isotopic to a horizontal surface via an ambient isotopy fixing $F_\pm$.

Remark 1.38. The second case also includes the possibility that S intersects neither F_+ nor F_-. In this situation S is ambient isotopic to $F \times \{t\}$ for $t \in (-1,1)$.

Later we need the following lemma, which easily follows from Waldhausen's result.

Lemma 1.39. *Let $N \ncong S^1 \times S^2$ be a connected oriented 3-manifold which fibres over S^1 with fibre F. We fix an embedding of F and an identification $N \backslash\backslash F \cong F \times [-1,1]$. If Σ is an incompressible surface in N such that $F \cap \Sigma$ has minimal number of connected components compared to all other embeddings in the isotopy class of Σ, then one of the following holds:*

1. *Σ consists of parallel copies of surfaces, all isotopic to F,*

2. *every component of $\Sigma' := \Sigma \cap N \backslash\backslash F$ intersects F_+ and F_-.*

Proof . A standard argument using the irreducibility of N and our hypothesis on $F \cap \Sigma$ shows, that $F \cap \Sigma$ is incompressible and hence Σ' is incompressible in $F \times [-1,1]$. If Σ' does not intersect $F_+ \cup F_-$, then by Remark 1.38 every component of Σ' is ambient isotopic to $F \times \{t\}$ for some $t \in (-1,1)$. Therefore, Σ consist of parallel copies of F. Now let C be connected component of Σ', which intersects F_- at least once but does not intersect F_+. Then there is an ambient isotopy fixing $F_\pm$, which makes C into a horizontal surface. Since this isotopy fixes $F_\pm$, this isotopy extends to an isotopy of Σ in N. If we further assume that C is an innermost among such a connected component, then one can use the isotopy from Remark 1.37 to remove the intersection component which corresponds to $C \cap F_-$. But this contradicts our assumptions on $F \cap \Sigma$ and hence C has to intersect F_+. The same argument with the roles of F_+ and F_- interchanged proves the lemma. □

Lemma 1.40. *Let* $(N, \emptyset, \emptyset, \partial N)$ *be a taut sutured manifold. If* Σ *is a quasi-fibre in* N *which is not the fibre of a fibration, then the sutured manifold* M *obtained by* $N \overset{\Sigma}{\rightsquigarrow} M$ *contains a decomposition surface* S *such that* $M \overset{S}{\rightsquigarrow} M'$ *results in a product sutured manifold* M' *and the class* $[S] \in H_2(M, \partial M; \mathbb{Z})$ *is non-trivial.*

Proof . Since Σ is a quasi-fibre there is a fibration of N over S^1 with fibre F and $F \oplus \Sigma$ is taut. We can assume that F and Σ are in general position such that the number of components of $\Sigma \cap F$ is minimal compared to all surfaces isotopic to Σ and F.

We define $F' := F \cap M = F \setminus \nu(F \cap \Sigma)$ and make the following claim.

Claim. The sutured decomposition $(M, \Sigma_+, \Sigma_-, \gamma) \overset{F'}{\rightsquigarrow} (M', R'_+, R'_-, \gamma')$ results in a product sutured manifold.

We set $\Sigma' = \Sigma \cap (N \backslash\backslash F)$. By Lemma 1.20 we have that M' is a taut sutured manifold and by the commutativity of the diagram in Lemma 1.20 we have that $N \overset{F}{\rightsquigarrow} N \backslash\backslash F \overset{\Sigma'}{\rightsquigarrow} M'$. Moreover, $N \backslash\backslash F \cong F \times [-1, 1]$ is a product, since F is a fibre of a fibration. The taut sutured manifold decomposition of a product sutured manifold is again a product sutured manifold (see [Ga83, Remark 4.9(4)]) and hence M' is a product.

It remains to show, that $[F'] \in H_2(M, \partial M; \mathbb{Z})$ is non-trivial. This follows directly from the next claim.

Claim. There is a closed curve c in N, which does not intersect Σ but has a positive intersection number with F.

The curve c is constructed as follows (see Figure 1.7). We choose a point $x_0 \in F_+ \backslash \Sigma'$ and a path p_0 not intersecting Σ' to F_-. Such a path always exists by Lemma 1.39 and the assumption that Σ is not a fibre of a fibration. The monodromy sends the endpoint of this path to a new point x'_1 on F_+ maybe in a different connected component of M'. We repeat this process to obtain another path p_1 connecting x'_1 with another point $x_2 \in F_-$, which is sent to $x'_2 \in F_+$ via the monodromy. Since there are only finitely many connected components of M' we can after several iterations of this process join x_n with x_0 in F_+ by a path p not intersecting Σ'. All these paths patched together give a closed loop in N. This loop does not intersect Σ but gives a positive intersection number with F.

Since c does not intersect Σ it is a loop in M and since it has positive intersection number with F, the class $[F'] \in H_2(M, \partial M; \mathbb{Z})$ is non-trivial. □

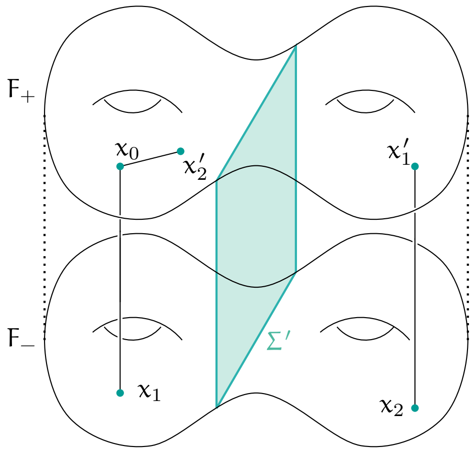

Figure 1.7.: A schematic picture of how the loop c in the proof of Lemma 1.40 is constructed. We see the product surface $F \times [-1, 1]$. The rectangle represents $\Sigma \cap F \times [-1, 1]$. The points x_i and x_i' are identified by the monodromy in N. The concatenation of the paths gives a loop in N.

The next proposition is implicitly already in an article of Agol [Ag08, Theorem 6.1]. We will show how it follows from Agol's virtual fibring theorem, which we first recall. Notice that the fundamental group of an irreducible 3-manifold with non-empty boundary is virtually RFRS [AFW15, Corollary 4.8.7], which was proved by Przytycki and Wise [PW17]. We don't need the precise definition of RFRS, we only need that the following theorem holds in our situation.

Theorem 1.41. *[Ag08, Theorem 5.1] Let M be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary. Assume $\pi_1(M)$ is infinite and virtually RFRS. If Σ is a taut surface, then there is a finite sheeted cover $p\colon N \to M$ such that $p^{-1}(\Sigma)$ is a quasi-fibre.*

Proposition 1.42. *Let (M, γ) be a connected taut sutured manifold with infinite fundamental group and such that γ is incompressible. Then there exists a connected finite cover $(\widehat{M}, \widehat{\gamma})$ and a decomposition surface S in $\widehat{M}$, such that $\widehat{M} \overset{S}{\rightsquigarrow} \widehat{M}'$ is a product sutured manifold and $[S] \in H_2(\widehat{M}, \partial\widehat{M})$ is non-trivial.*

Proof . If (M, γ) is a product sutured manifold, then one can take the annulus obtained from a homologically non-trivial curve in R_- times the interval. Therefore, we will in the following assume that (M, γ) is a taut sutured manifold which is not a product. We consider the double $DM(\gamma)$ of a sutured manifold M:

$$DM(\gamma) := M \sqcup_{R_\pm} M.$$

This is a 3-manifold, which is closed or has toroidal boundary. Since γ is incompressible and (M, γ) is taut by our assumptions, we have that $R_- \cup R_+$ is a taut surface in $DM(\gamma)$ [Ga83, Lemma 3.7].

By Theorem 1.41 we obtain a finite cover $p\colon W \to DM(\gamma)$ such that the surface $p^{-1}(R_- \cup R_+)$ is a quasi-fibre. We write $\Sigma = p^{-1}(R_- \cup R_+)$. The taut sutured manifold W' given by $W \overset{\Sigma}{\rightsquigarrow} W'$ is by construction a finite cover of M. Therefore, by Lemma 1.40 applied to Σ and W we see that a connected component $\widehat{M} \subset W'$ is the desired finite cover of M. $\square$

CHAPTER 2

L^2-invariants

In this chapter we introduce some L^2-invariants. Most of the parts here are taken from the monograph of Lück [Lü02]. The purpose of this chapter is to fix our notation and hence we omit most proofs.

2.1. $\mathcal{N}(G)$-modules and the von Neumann dimension

Let G be a countable group. We denote by $\mathbb{C}[G]$ the group ring of G. We define a scalar product on $\mathbb{C}[G]$ by linear extension of $\langle g_i, g_j\rangle = \delta_{i,j}$ for all $g_i, g_j \in G$. This makes $\mathbb{C}[G]$ into a pre-Hilbert space. We denote by $L^2(G)$ the Hilbert space completion of the group ring $\mathbb{C}[G]$. One easily verifies:

$$L^2(G) = \left\{ \sum_{g\in G} a_g \cdot g \mid a_g \in \mathbb{C},\ \sum_{g\in G} |a_g|^2 < \infty \right\}.$$

Let $\mathcal{B}(L^2(G))$ be the set of linear bounded operator from $L^2(G)$ to itself. The standard left action of G on $\mathbb{C}[G]$ extends to an isometric action on $L^2(G)$ and we can consider the operator that commute with this action. We define the group von Neumann algebra by

$$\mathcal{N}(G) := \left\{ A \in \mathcal{B}(L^2(G)) \mid g(Ax) = A(gx) \text{ for all } g \in G \text{ and } x \in L^2(G) \right\}.$$

The group von Neumann algebra comes with a trace defined by

$$\begin{aligned} \operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) &\longrightarrow \mathbb{C} \\ f &\longmapsto \langle f(e), e\rangle. \end{aligned}$$

Here e denotes the neutral element in G. More generally if $A = (a_{ij})_{\substack{i\in\{1,\dots,n\}\\ j\in\{1,\dots,n\}}}$ is a matrix in $\mathrm{Mat}_{n\times n}(\mathcal{N}(G))$, then we define

$$\operatorname{tr}_{\mathcal{N}(G)}(A) := \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(a_{ii}).$$

In the following we will define a „nice" dimension function for modules over $\mathcal{N}(G)$. Let P be a finitely generated projective module over $\mathcal{N}(G)$. There is a module S

such that $\mathcal{N}(G)^n \cong P \oplus S$. Choose a matrix $A \in \mathrm{Mat}_{n\times n}(\mathcal{N}(G))$ with $A^2 = A$ and $\operatorname{Im} A \cong P$. We define the von Neumann dimension of P by

$$\dim_{\mathcal{N}(G)} P \coloneqq \mathrm{tr}_{\mathcal{N}(G)} A.$$

Lemma 2.1. *For a finitely generated projective $\mathcal{N}(G)$-module P, the dimension $\dim_{\mathcal{N}(G)} P$ is well-defined i.e. independent of the choice of A.*

Proof . The proof is a standard argument from linear algebra. Suppose there is another matrix $B \in \mathrm{Mat}_{p\times p}(\mathcal{N}(G))$ with $B^2 = B$ and $\operatorname{Im} B \cong P$. We can assume that $p = n$ by filling up one matrix with zeros which does not change the trace and the isomorphism class of the image. Since A and B are projection matrices we have $\mathcal{N}(G)^n \cong \operatorname{Im} A \oplus \operatorname{Im}(\mathrm{Id} - A)$ and $\mathcal{N}(G)^n \cong \operatorname{Im} B \oplus \operatorname{Im}(\mathrm{Id} - B)$. The abstract isomorphism $\operatorname{Im} A \cong \operatorname{Im} B$ can be used to find an invertible $C \in \mathrm{Mat}_{n\times n}(\mathcal{N}(G))$ such that $CAC^{-1} = B$. Note that $\mathrm{tr}_{\mathcal{N}(G)}$ satisfies $\mathrm{tr}_{\mathcal{N}(G)}(DE) = \mathrm{tr}_{\mathcal{N}(G)}(ED)$ for arbitrary $E, D \in \mathrm{Mat}_{n\times n}(\mathcal{N}(G))$. Hence we conclude

$$\mathrm{tr}_{\mathcal{N}(G)} B = \mathrm{tr}_{\mathcal{N}(G)} CAC^{-1} = \mathrm{tr}_{\mathcal{N}(G)} CC^{-1}A = \mathrm{tr}_{\mathcal{N}(G)} A. \qquad \square$$

Given an arbitrary $\mathcal{N}(G)$-module M, we extend the definition of the von Neumann dimension by

$$\dim_{\mathcal{N}(G)} M \coloneqq \sup \left\{ \dim_{\mathcal{N}(G)} P \mid P \subset M \text{ finetely genereted projective submodule} \right\}.$$

Before we summarise the basic properties we introduce some more notation.

Definition 2.2. Let M be a $\mathcal{N}(G)$-submodule of N. We write N^* for the dual space of N. We define the closure of M in N by

$$\overline{M} \coloneqq \{x \in N \mid f(x) = 0 \text{ for all } f \in N^* \text{ with } M \subset \ker f \text{ .}\}$$

Moreover, we call for any $\mathcal{N}(G)$-module M the closure of the trivial submodule $\{0\}$ the torsion part of M and we write

$$\mathbf{T}M \coloneqq \overline{\{0\}},$$
$$\mathbf{P}M \coloneqq M/\mathbf{T}M,$$

and refer to $\mathbf{P}M$ as the projective part of M.

We say that a map $f \colon M \to W$ between two $\mathcal{N}(G)$-modules has dense image if $\overline{\operatorname{Im} f} = W$. If f is injective and has dense image, then we call f a weak isomorphism.

We summarise the results we need in the following theorem. A proof can be found in Lück's book on L^2-invariants [Lü02, Chapter 6].

Theorem 2.3. *1. If M is finitely generated $\mathcal{N}(G)$-module, then*

$$\dim_{\mathcal{N}(G)} M = \dim_{\mathcal{N}(G)}(\mathbf{P}M),$$
$$\dim_{\mathcal{N}(G)} \mathbf{T}M = 0.$$

2. *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is an exact sequence of* $\mathcal{N}(G)$*-modules, then*
$$\dim_{\mathcal{N}(G)} M_2 = \dim_{\mathcal{N}(G)} M_1 + \dim_{\mathcal{N}(G)} M_3$$
with the usual convention that $r + \infty = \infty$.

3. *Let* $f\colon P \to Q$ *be an* $\mathcal{N}(G)$*-morphism of finitely generated projective* $\mathcal{N}(G)$*-modules. If* f *is a weak isomorphism, then* P *and* Q *are* $\mathcal{N}(G)$*-isomorphic.*

4. *If* $f\colon P \to Q$ *is an* $\mathcal{N}(G)$*-morphism of finitely generated projective* $\mathcal{N}(G)$*-modules with* $\dim_{\mathcal{N}(G)} P = \dim_{\mathcal{N}(G)} Q$*, then the following assertions are equivalent:*
   a) f *is injective,*
   b) f *has dense image,*
   c) f *is a weak isomorphism.*

5. *A projective* $\mathcal{N}(G)$*-module* P *is trivial if and only if* $\dim_{\mathcal{N}(G)}(P) = 0$.

6. *Let* H *be a finite index subgroup of* G *and* M *a* $\mathcal{N}(G)$*-module, then* M *can be seen as a* $\mathcal{N}(H)$*-module by restricting the action. In this case one has*
$$\dim_{\mathcal{N}(H)} M = [G : H] \cdot \dim_{\mathcal{N}(G)} M.$$

7. *If* M *is a left* $\mathcal{N}(H)$*-module, then* $\mathcal{N}(G) \otimes_{\mathcal{N}(H)} M$ *is a left* $\mathcal{N}(G)$*-module and one has*
$$\dim_{\mathcal{N}(G)} \mathcal{N}(G) \otimes_{\mathcal{N}(H)} M = \dim_{\mathcal{N}(H)} M.$$

8. *Let* $\{M_i \mid i \in I\}$ *be a cofinal system of* $\mathcal{N}(G)$*-submodules of* M *with* $\cup_{i\in I} M_i = M$*. One has*
$$\dim_{\mathcal{N}(G)} M = \sup_{i\in I} \left\{\dim_{\mathcal{N}(G)} M_i\right\}.$$

Remark 2.4. Let $V \subset L^2(G)$ be a G-equivariant Hilbert subspace. Then one can take a matrix $P \in \mathrm{Mat}_{n\times n}(\mathcal{N}(G))$ with $P = P^*$, $P^2 = P$, and $\mathrm{Im}\, P = V$ and define a dimension $\dim_{\mathcal{N}G} V \coloneqq \mathrm{tr}_{\mathcal{N}(G)} P$.

This dimension satisfies similar properties as the dimension defined above for $\mathcal{N}(G)$-modules. More generally there is an equivalence of categories of finitely generated projective $\mathcal{N}(G)$-modules and subspaces described above. This equivalence preserves the dimension [Lü02, Theorem 6.24].

Since the dimension of an $\mathcal{N}(G)$-module M only depends on the projective part $\mathbf{P}M$, it can be sometimes useful to have long exact sequence disregarding the torsion part.

Lemma 2.5. *Let* $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$ *be a short exact sequence of finitely generated free* $\mathcal{N}(G)$*-chain complexes. Then there is a long exact sequence*
$$\ldots \longrightarrow \mathbf{P}\, H_i(C_*) \longrightarrow \mathbf{P}\, H_i(D_*) \longrightarrow \mathbf{P}\, H_i(E_*) \longrightarrow \ldots.$$

Proof . This is a consequence of Lemma 6.52 and Theorem 1.21 in Lück's book [Lü02]. □

2.2. L^2-Betti numbers

In this section we introduce L^2-Betti numbers. With the preceding section and the theory of twisted coefficients they are easily defined. If X is a connected CW-complex with fundamental group π, then $\mathcal{N}(\pi)$ is a right $\mathbb{Z}[\pi]$-module. Therefore, we have the twisted homology as defined in Appendix B:

$$H_i(X; \mathcal{N}(\pi)) \coloneqq H_i\left(\mathcal{N}(\pi) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})\right).$$

Definition 2.6 (L^2-Betti numbers)**.** Let X be a connected CW-complex. The L^2-Betti number is defined by

$$b_i(X; \mathcal{N}(\pi)) := \dim_{\mathcal{N}(\pi)} H_i(X; \mathcal{N}(\pi)).$$

More generally, let $\widehat{X}$ be a topological space on which a group G acts properly discontinuous from the left and let $\widehat{Y} \subset \widehat{X}$ be a G-invariant subspace. We write $X \coloneqq G\backslash\widehat{X}$ and $Y \coloneqq G\backslash\widehat{Y}$ for the quotient spaces. We obtain a chain complex

$$C_*(X, Y; \mathcal{N}(G)) := \mathcal{N}(G) \otimes_{\mathbb{Z}[G]} C_*(\widehat{X}, \widehat{Y})$$

and define the L^2-homology as the homology of the chain complex

$$H_i(X, Y; \mathcal{N}(G)) \coloneqq H_i\left(\mathcal{N}(G) \otimes_{\mathbb{Z}[G]} C_*(\widehat{X}, \widehat{Y})\right).$$

We obtain relative L^2-Betti numbers as well by

$$b_i(X, Y; \mathcal{N}(G)) \coloneqq \dim_{\mathcal{N}(G)} H_i(X, Y; \mathcal{N}(G)).$$

Sometimes it is convenient that one can calculate L^2-Betti numbers using a cellular chain complex. Therefore, let $\widehat{X}$ be a CW-complex on which a group G acts cocompactly, freely and cellularly and $\widehat{Y}$ a G-invariant subcomplex. We can consider the chain complex

$$C_*^{cw}(X, Y; \mathcal{N}(G)) := \mathcal{N}(G) \otimes_{\mathbb{Z}[G]} C_*^{CW}(\widehat{X}, \widehat{Y})$$

with the cellular L^2-homology defined by

$$H_i^{cw}(X, Y; \mathcal{N}(G)) := H_i\left(C_*^{cw}(X, Y; \mathcal{N}(G))\right).$$

We have following theorem which relates these two concepts.

Theorem 2.7. *[Lü02, Lemma 6.51] There is an isomorphism between* $H_i^{cw}(X, Y; \mathcal{N}(G))$ *and* $H_i(X, Y; \mathcal{N}(G))$.

Hence we can calculate the L^2-Betti numbers using a cellular chain complex. We can also consider L^2-cohomology. Regarding the L^2-Betti number we have the following version of the universal coefficient theorem [Lü02, Lemma 1.18]:

$$\dim_{\mathcal{N}(G)} H_i^{cw}(X, Y; \mathcal{N}(G)) = \dim_{\mathcal{N}(G)} H^i_{cw}(X, Y; \mathcal{N}(G)). \tag{2.1}$$

If X is a finite CW-complex, then we define the L^2-Euler characteristic by

$$\chi(X; \mathcal{N}(G)) := \sum_{i \in \mathbb{N}_0} (-1)^i \cdot b_i(X; \mathcal{N}(G)).$$

Remark 2.8. These numbers only depend on the homotopy type of $\widehat{X}/G$ and are especially independent of the choice of CW-structure.

By the additivity of the von Neumann dimension (Theorem 2.3 (2)) one can show that

$$\chi(X; \mathcal{N}(G)) = \sum_{i \in \mathbb{N}_0} (-1)^i \#\ i\text{-cells} = \chi(X). \tag{2.2}$$

One of the main features of L^2-Betti numbers is given by the next proposition, which is a direct consequence of Theorem 2.3 (6).

Proposition 2.9. *Let $\widehat{X}$ be a CW-complex on which a group G acts co-compactly, freely and cellularly and $\widehat{Y} \subset \widehat{X}$ be a G-invariant subspace. If $H \triangleleft G$ is a finite index subgroup, then we have*

$$b_i(H\backslash\widehat{X}, H\backslash\widehat{Y}; \mathcal{N}(H)) = [G : H] \cdot b_i(G\backslash\widehat{X}, G\backslash\widehat{Y}; \mathcal{N}(G)).$$

In this thesis we are more interested in the L^2-Betti numbers rather than the concrete structure of the homology groups. For this purpose one should recall the definition of the projective part of an $\mathcal{N}(\pi)$-module (see Definition 2.2). In this context we introduce the following notation. Given a connected CW-complex X with fundamental group π, we write $p\colon \widetilde{X} \to X$ for the corresponding universal cover and we define the projective L^2-homology of X by:

$$H_i^{(2)}(X) := \mathbf{P}\, H_i(\widetilde{X}; \mathcal{N}(\pi)).$$

We set

$$b_i^{(2)}(X) = \dim_{\mathcal{N}(\pi)} H_i^{(2)}(X).$$

Obviously, by Theorem 2.3 (1) we have $b_i^{(2)}(X) = b_i(X; \mathcal{N}(\pi))$. In the case that X has several connected components $X_1, \ldots, X_n$ we put $b_i^{(2)}(X) := \sum_{k=1}^n b_i^{(2)}(X_k)$.

Moreover, if $Y \subset X$ is a subcomplex, then we write $\widetilde{Y} := p^{-1}(Y)$ and define

$$H_i^{(2)}(Y \subset X) := \mathbf{P}\, H_i(\widetilde{Y}; \mathcal{N}(\pi)), \quad b_i^{(2)}(Y \subset X) := \dim_{\mathcal{N}(\pi)} H_i^{(2)}(Y \subset X),$$
$$H_i^{(2)}(X, Y) := \mathbf{P}\, H_i(\widetilde{X}, \widetilde{Y}; \mathcal{N}(\pi)), \quad b_i^{(2)}(X, Y) := \dim_{\mathcal{N}(\pi)} H_i^{(2)}(X, Y).$$

Using Theorem 2.5 one has a Mayer-Vietoris sequences and the long exact sequence associated to a pair. Moreover, if the inclusion $Y \to X$ induces for any choice of base point a monomorphism on the fundamental group, then one has by the induction principle (Theorem 2.3 (7))

$$b_i^{(2)}(Y \subset X) = b_i^{(2)}(Y). \tag{2.3}$$

We consider another abbreviation. If $\phi \colon \pi \to \mathbb{Z}$ is an epimorphism, then the corresponding cover to $\ker \phi$ is written $X_{\ker \phi}$. In this case $\mathbb{Z}$ acts via deck transformation on $X_{\ker \phi}$ and we introduce the notation

$$H_*^{\phi,(2)}(X) := \mathbf{P}\, H_*(X; \mathcal{N}(\mathbb{Z})).$$

Let $\langle t \rangle \cong \mathbb{Z} \cong \pi/\ker \phi$ denote a generator. One has [Lü02, Lemma 1.34]:

$$\dim_{\mathcal{N}(\mathbb{Z})} H_i^{\phi,(2)}(X) = \dim_{\mathbb{C}(t)} H_i(X; \mathbb{C}(t)^{\phi}), \tag{2.4}$$

where we endow $\mathbb{C}(t)$ with a $\mathbb{Z}[\pi]$-module structure using ϕ. To be more precise for $g \in \pi$ and $p(t) \in \mathbb{C}(t)$ we define a left action by $g \cdot p(t) := t^{\phi(g)} \cdot p(t)$.

Remark 2.10. If $\phi \colon \pi \to \mathbb{Z}$ is an epimorphism, then one has an isomorphism:

$$H_i(X; \mathbb{C}(t)^{\phi}) \cong \mathbb{C}(t) \otimes_{\mathbb{Z}[t^{\pm}]} H_i(X_{\ker \phi}; \mathbb{Z}).$$

Suppose that $\phi \colon \pi \to \mathbb{Z}$ has infinitive image but is not an epimorphism. One still can view $\mathbb{C}(t)$ as an module over $\mathbb{Z}[\pi]$ using ϕ. Hence it makes sense to consider the twisted homology groups $H_i(X; \mathbb{C}(t)^{\phi})$. It turns out that one has the equality of dimensions:

$$\dim_{\mathbb{C}(t)} H_i(X; \mathbb{C}(t)^{\phi}) = \dim_{\mathbb{C}(t)} \mathbb{C}(t) \otimes_{\mathbb{Z}[t^{\pm}]} H_i(X_{\ker \phi}; \mathbb{Z}).$$

Approximation of L^2-Betti numbers

In this paragraph we recall the Lück-Schick approximation result of L^2-Betti numbers. In order to state the theorem and that it applies in our situation we need some preliminaries.

Definition 2.11. Let $\mathcal{G}$ be the smallest class of groups which contains the trivial group and is closed under the following processes:

1. If $H < \pi$ is a normal subgroup such that $H \in \mathcal{G}$ and π/H is amenable, then $\pi \in \mathcal{G}$.
2. If π is the direct limit of a directed system of groups $\pi_i \in \mathcal{G}$, then $\pi \in \mathcal{G}$.
3. If π is the inverse limit of a directed system of groups $\pi_i \in \mathcal{G}$, then $\pi \in \mathcal{G}$.
4. The class $\mathcal{G}$ is closed under taking subgroups.

The precise definition of an amenable group doesn't play a role for this thesis. We only need that finite groups are amenable. This is sufficient to prove the following lemma.

Lemma 2.12. *Every fundamental group of a compact connected 3-manifold lies in $\mathcal{G}$.*

Proof . By fact (1) every finite group lies in $\mathcal{G}$. Then by fact (3) the profinite completion of a group lies in $\mathcal{G}$. Since residually finite groups are subgroups of their profinite completion we have by fact (4) that all residually finite groups are in $\mathcal{G}$. So the lemma follows from the fact that all 3-manifold groups are residually finite [AFW15, Chapter 3 C.29]. □

We are now able to state the approximation result of Schick which extended earlier results by Lück [Lü94].

Theorem 2.13. *[Sch01, Theorem 1.14] Let $\widehat{X}$ be a CW-complex on which a group G acts co-compactly, freely and cellularly and $\widehat{Y} \subset \widehat{X}$ be a G-invariant subspace. We write $X = G\backslash\widehat{X}$ and $Y = G\backslash\widehat{Y}$. Let $G = G_1 \supset G_2 \supset \dots$ be a nested sequence of normal subgroups such that $\cap_{i\in\mathbb{N}} G_i = \{e\}$. Further assume that $G \in \mathcal{G}$ and $G/G_i \in \mathcal{G}$ for all $i \in \mathbb{N}$. One has for all $p \in \mathbb{Z}$:*

$$\lim_{i\to\infty} b_p(X, Y; \mathcal{N}(G/G_i)) = b_p(X, Y; \mathcal{N}(G)).$$

Remark 2.14. A simple consequence of the approximation result is the following observation. If G is infinite, then $b_0(X; \mathcal{N}(G)) = 0$. However, this can be proven directly without the technical assumptions on G [Lü02, Theorem 1.35 (8)].

2.3. L^2-torsion

Fuglede Kadison-determinant

In this section we give the definition of the Fuglede-Kadison determinant. Given a matrix $A \in Mat_{n\times m}(\mathcal{N}(G))$, we can view it as a map $f_A\colon L^2(G)^n \to L^2(G)^m$ by acting from the right. We denote by $L(f_A, \lambda)$ the set of all subspaces $L \subset L^2(G)^n$ such that $\|f_A(x)\| \leqslant \lambda\|x\|$ for all $x \in L$. We define the spectral density function

$$\begin{aligned} F(A) : \mathbb{R}^+ &\to \mathbb{R}^+ \\ \lambda &\longmapsto \sup\left\{\dim_{\mathcal{N}(G)} L \mid L \in L(f_A, \lambda)\right\}. \end{aligned}$$

In Lück's book [Lü02, Chapter 3.2] it is proven that this function is right continues and hence defines a Borel measure by $F(A)((a, b]) = F(A)(b) - F(A)(a)$. The Fuglede-Kadison determinant is defined by:

$$\det_{\mathcal{N}(G)}(A) := \begin{cases} \exp\left(\int_{0^+}^{\infty} \ln(\lambda) dF(A)\right), & \int_{0^+}^{\infty} \ln(\lambda) dF > -\infty, \\ 0, & \text{else.} \end{cases}$$

We say that a matrix $A \in \mathrm{Mat}_{n\times m}(\mathcal{N}(G))$ is of determinant class if $\det_{\mathcal{N}(G)} A \neq 0$. One property we often use is the so called induction principle, which is a consequence of Theorem 2.3 (7).

Lemma 2.15. *Let* H *be a subgroup of* G *and* $A \in \mathrm{Mat}_{n\times m}(\mathcal{N}(H))$*. We can view* A *as a matrix in* $\mathrm{Mat}_{n\times m}(\mathcal{N}(G))$*, which we denote by* i_*A*. If* A *is of determinant class, then* i_*A *is of determinant class and we have:*

$$\det_{\mathcal{N}(G)} i_*A = \det_{\mathcal{N}(H)} A.$$

Another useful property is that one can calculate the determinant via a finite index subgroup. To be more precise we have the following lemma.

Lemma 2.16. *Let* H *be a finite index subgroup of* G *and* $A \in \mathrm{Mat}_{n\times m}(\mathcal{N}(G))$*. By choosing a full set of cosets* $g_1 \cdot H, \dots, g_k \cdot H$*, we obtain a matrix* $\operatorname{res} A \in \mathrm{Mat}_{kn\times km}(\mathcal{N}(G))$*. If* A *is of determinant class, then* $\operatorname{res} A$ *is of determinant class and we have:*

$$\det_{\mathcal{N}(H)} \operatorname{res} A = \left(\det_{\mathcal{N}(G)} A\right)^k.$$

In particular, the value of $\det_{\mathcal{N}(H)} \operatorname{res} A$ *does not depend on the choice of cosets.*

If one compares the Fuglede-Kadison determinant with the classical determinant for vector spaces, then it might seem odd that it is defined even for non-square matrices. This leads also to strange behaviour concerning limits.

Example 2.17. If we view $\mathbb{C}$ as the group von Neumann algebra of the trivial group $\{e\}$, then for the sequence of 1×1-matrices $1/n$, we have $\lim_{n\to\infty} \det_{\mathcal{N}(\{e\})} 1/n = 0$, but $\det_{\mathcal{N}(\{e\})} 0 = 1$.

For this reason we follow Dubois, Friedl and Lück [DFL16] and consider the regularised Fuglede-Kadison determinant. Let $A \in \mathrm{Mat}_{n\times n}(\mathcal{N}(G))$ be a matrix. We define the regularised Fuglede-Kadison determinant by

$$\det^{r}_{\mathcal{N}(G)} A := \begin{cases} \det_{\mathcal{N}(G)} A & \text{if } A \text{ is a weak isomorphism,} \\ 0 & \text{else.} \end{cases}$$

For the regularised Fuglede-Kadison determinant we have the following two properties that are similar to the classical determinant. They also hold for the Fuglede-Kadison determinant but with some extra assumptions.

Proposition 2.18. *Given matrices* $A \in \mathrm{Mat}_{n\times n}(\mathcal{N}(G))$, $B \in \mathrm{Mat}_{k\times k}(\mathcal{N}(G))$*, and* $C \in \mathrm{Mat}_{n\times k}(\mathcal{N}(G))$*, one has*

$$\det^{r}_{\mathcal{N}(G)} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det^{r}_{\mathcal{N}(G)} A \cdot \det^{r}_{\mathcal{N}(G)} B.$$

Moreover, if $k = n$*, then*

$$\det^{r}_{\mathcal{N}(G)} AB = \det^{r}_{\mathcal{N}(G)} A \cdot \det^{r}_{\mathcal{N}(G)} B.$$

In particular, the regularised Fuglede-Kadison determinant does not change under elementary row and column operations.

We later need one particular calculation [Lü02, Example 3.13]:

Lemma 2.19. *Let G be a group and $\gamma \in G$ be an element. If γ is not the neutral element, then for any $t \in \mathbb{R}_{>0}$ and any $w \in \mathbb{N}$ we have $\det^{\mathbf{r}}_{\mathcal{N}(G)}(1 - t^w \cdot \gamma) = \max\{1, t^w\}$.*

Another feature of the regularised Fuglede-Kadison determinant is the following lemma proven by Liu.

Lemma 2.20 ([Li17, Lemma 3.1])**.** *If a sequence $\{A_k\}_{k\in\mathbb{N}}$ of matrices over $\mathcal{N}(G)$ converges to a matrix $A \in \mathrm{Mat}_n(\mathcal{N}(G))$ in the norm topology, then*

$$\limsup_{k\to\infty} \ \det^{\mathbf{r}}_{\mathcal{N}(G)} A_k \leqslant \det^{\mathbf{r}}_{\mathcal{N}(G)} A.$$

L^2-torsion of a finite based chain complex

Next we define the L^2-torsion of a chain complex using the Fuglede-Kadison determinant. But first we fix some notation. In the following we call a $\mathcal{N}(G)$-chain complex C_* finite if $C_i = 0$ for all but finitely many $i \in \mathbb{Z}$ and based with bases $\mathcal{C}$ if for each $i \in \mathbb{Z}$ we have a preferred isomorphism $C_i \cong \mathcal{N}(G)^{r_i}$.

Given a finite based chain complex $(C_*, \partial_*, \mathcal{C})$, one can use the preferred isomorphism to identify the boundary map ∂_i with a matrix $A_i \in \mathrm{Mat}_{r_i \times r_{i-1}}(\mathcal{N}(G))$ multiplying from the right.

Suppose that every A_i is of determinant class. We define the L^2-torsion of the chain complex $(C_*, \partial_*^C, \mathcal{C})$ to be

$$\tau^{(2)}(C_*, \partial_*^C, \mathcal{C}) := \prod_{i\in\mathbb{Z}} \det_{\mathcal{N}(G)} A_i.$$

Definition 2.21. (L^2-det-acyclic) A finite based chain complex $(C_*, \partial_*^C, \mathcal{C})$ is called L^2-det-acyclic if $PH_i(C_*) = 0$ and ∂_i is of determinant class for all $i \in \mathbb{N}$.

Definition 2.22. Let $(C_*, \partial_*^C, \mathcal{C})$, $(D_*, \partial_*^D, \mathcal{D})$ and $(E_*, \partial_*^E, \mathcal{E})$ be finite based $\mathcal{N}(G)$-chain complexes. Given a short exact sequence $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$, we say that the chain complexes have compatible bases if for all $k \in \mathbb{Z}$ one has that $\det_{\mathcal{N}(G)} i_k = 1$ and $\det_{\mathcal{N}(G)} p_k = 1$.

One of the most useful property of the L^2-torsion is the following lemma.

Lemma 2.23. *[Lü02, Theorem 3.35] Let $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$ be a short exact sequence of finite based chain complexes with compatible basis. If two out of three are L^2-det-acyclic, then the third is L^2-det-acyclic and one has*

$$\tau^{(2)}(C_*, \partial_*^C, \mathcal{C}) \cdot \tau^{(2)}(E_*, \partial_*^E, \mathcal{E}) = \tau^{(2)}(D_*, \partial_*^D, \mathcal{D}).$$

We briefly mention a version of this lemma for not necessarily acyclic chain complexes. This version will later motivate some results and questions.

A short exact sequence of chain complexes $0 \to C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \to 0$ give rise to a long exact sequence in homology $\ldots \to H_k(C_*) \xrightarrow{i} H_k(D_*) \xrightarrow{p} H_k(E_*) \to \ldots$. In Lück's book [Lü02, Chapter 3] is described an L^2-torsion where one views the long exact sequence as a based chain complex $\mathrm{LHS}_*(C_*, D_*, E_*)$. In this context one has the following extended version of the previous lemma.

Lemma 2.24. *Let $0 \to C_* \xrightarrow{i} D_* \xrightarrow{p} E_* \to 0$ be short exact sequence of finite based chain complexes with compatible basis. If all are of determinant class, then one has*

$$\tau^{(2)}(C_*, \partial_*^C, \mathcal{C}) \cdot \tau^{(2)}(E_*, \partial_*^E, \mathcal{E}) = \tau^{(2)}(D_*, \partial_*^D, \mathcal{D}) \cdot \tau^{(2)}(\mathrm{LHS}_*(C_*, D_*, E_*)).$$

L^2-Torsion of a CW-complex

Given a finite connected CW-complex X with fundamental group π, the cellular chain complex $C_*^{cw}(X; \mathcal{N}(\pi))$ is a finite free $\mathcal{N}(G)$-chain complex. For every cell c of X we pick a lift $\widetilde{c}$ to the universal cover $\widetilde{X}$. This defines a basis $\mathcal{C}$ of $C_*^{cw}(X; \mathcal{N}(\pi))$. We define the L^2-torsion of X and $\mathcal{C}$ to be

$$\tau^{(2)}(X, \mathcal{C}) := \tau^{(2)}(C_*^{cw}(X; \mathcal{N}(\pi)), \mathcal{C}).$$

We have the following result about the invariance of this number.

Theorem 2.25. *[Sch01, Theorem 1.14] If X is a finite connected CW-complex with $b_i^{(2)}(X) = 0$ for all $i \in \mathbb{N}$. If $\pi_1(X, x_0)$ is of class $\mathcal{G}$, then $\tau^{(2)}(X, \mathcal{C})$ is invariant under cellular homotopy equivalences and invariant under the choice of lifts. In particular, it does not depend on the choice of CW-structure.*

Remark 2.26. If we are in the above situation, then we suppress the choice of basis from the notation and just write $\tau^{(2)}(X)$.

We already discussed that for a connected 3-manifold the fundamental group lies in the class $\mathcal{G}$.

If Y is a subcomplex of X, then we define $\tau^{(2)}(X, Y)$ and $\tau^{(2)}(Y \subset X)$ as the L^2-torsion of the chain complexes $C_*^{cw}(X, Y; \mathcal{N}(\pi))$ and $C^{cw}(Y \subset X; \mathcal{N}(\pi))$. Lemma 2.15 in this context can be reformulated to the following lemma.

Lemma 2.27. *Let $Y \subset X$ be subcomplex of a connected finite CW-complex X. Suppose that $\pi_1(X)$ lies in the class of groups $\mathcal{G}$. If the inclusion induces a monomorphism on the fundamental group $\pi_1(Y, y) \to \pi_1(X, y)$ for any choice of base point y, then*

$$\tau^{(2)}(Y \subset X) = \tau^{(2)}(Y),$$

where $\tau^{(2)}(Y) := \prod_i^n \tau^{(2)}(Y_i)$ and $Y_1, \ldots Y_n$ are the connected components of Y.

Lemma 2.23 translates immediately to topology. We fix this in the next two lemmas.

Lemma 2.28. *Let* X *be a finite connected CW-complex and* $p\colon \widetilde{X} \to X$ *its universal cover. Let* $Y \subset X$ *be a subcomplex. Suppose that* $\pi_1(X)$ *lies in the class of groups* $\mathcal{G}$ *and for every* $y \in Y$ *we have that* $\pi_1(Y, y) \to \pi_1(X, y)$ *is injective. Then the following formula holds:*

$$\tau^{(2)}(X, Y) \cdot \tau^{(2)}(Y) = \tau^{(2)}(X)$$

Lemma 2.29. *Suppose that the* L^2*-Betti numbers of* (M, N) *vanish. Furthermore, suppose that* $(M, N) = (X \cup Y, C \cup D)$ *where* X *and* Y *are submanifolds such that each component of* $X \cap Y$ *is a submanifold of* ∂X *and* ∂Y *and the same holds for* (M, X, Y) *replaced by* (N, C, D)*. If for each component* Z *of* $X \cap Y$ *the* L^2*-Betti numbers of* $(Z, Z \cap N)$ *vanish and if the induced maps* $\pi_1(Z) \to \pi_1(X)$ *and* $\pi_1(Z) \to \pi_1(Y)$ *are monomorphisms, then*

$$\tau^{(2)}(M, N) = \tau^{(2)}(X, C) \cdot \tau^{(2)}(Y, D) \cdot \tau^{(2)}(X \cap Y, C \cap D)^{-1}.$$

There is a way to compute the torsion of a short chain complex. The proof is the same as for Reidemeister torsion an we refer to Turaev's book [Tu01, Theorem 2.2].

Lemma 2.30. *Let* G *be a group,* j, k, l *integers such that* $j < k$ *and* A, B, C *matrices with entries in* $\mathbb{C}[G]$ *of the respective sizes* $(k + l - j) \times l$*,* $k \times (k + l - j)$ *and* $j \times k$*. We consider the complex*

$$C_*\colon 0 \longrightarrow \mathcal{N}(G)^j \xrightarrow{C} \mathcal{N}(G)^k \xrightarrow{B} \mathcal{N}(G)^{k+l-j} \xrightarrow{A} \mathcal{N}(G)^l \longrightarrow 0.$$

Let $L \subset \{1, \dots, k + l - j\}$ *be a subset of size* l *and* $J \subset \{1, \dots k\}$ *a subset of size* j*. We write*

$$\begin{aligned} A[, L] &:= \textit{rows in A corresponding to } L, \\ B[\widehat{L}, \widehat{J}] &:= \textit{result of deleting the columns of } B \textit{ corresponding to } L \\ & \textit{and deleting the rows corresponding to J}, \\ C[J,] &:= \textit{columns of C corresponding to } J. \end{aligned}$$

If $\det^r_G(A[, L]) \neq 0$ *and* $\det^r_G(C[J,]) \neq 0$*, then*

$$\tau^{(2)}(C_*) = \det\nolimits^r_{\mathcal{N}(G)}(A[, L])^{-1} \cdot \det\nolimits^r_{\mathcal{N}(G)}(B[\widehat{L}, \widehat{J}]) \cdot \det\nolimits^r_{\mathcal{N}(G)}(C[J,])^{-1}.$$

L^2-invariants of an irreducible 3-manifold

We end this section with the calculation of the L^2-torsion for an irreducible 3-manifold. We recall the geometric decomposition stated in the introduction of this thesis.

Theorem 2.31. *[AFW15, Theorem 1.7.6] Let N be a compact orientable irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \ldots, T_m$ in N such that each component of N cut along $T_1 \cup \ldots \cup T_m$ is hyperbolic or Seifert fibred. Furthermore, any such collection of tori with a minimal number of components is unique up to isotopy*

With this theorem we can calculate the L^2-invariants using the gluing formulas and reduce the calculations to the cases of Seifert fibred space and hyperbolic 3-manifold of finite volume. For these calculations we need the following vanishing result of the L^2-torsion for spaces with certain S^1-actions.

Lemma 2.32. *[Lü02, Theorem 3.105] Let M be a subcomplex of a finite CW-complex X. If M is a principle S^1-bundle over a finite CW-complex such that for any fibre F the inclusion $\pi_1(F) \to \pi_1(M) \to \pi_1(X)$ is a monomorphism, then $b_*^{(2)}(M \subset X) = 0$. Moreover, the L^2-torsion is well-defined and one has $\tau^{(2)}(M \subset X) = 1$.*

In particular, we have for a torus: $\tau^{(2)}(T^2) = 1$. Now it is a classical result that if M is a Seifert fibred space, then there is a finite cover $M' \to M$, such that M' is a principle S^1-bundle [AFW15, Section 3.2(C.10)]. Hence, if M has infinite fundamental group, then we have by Proposition 2.9 that

$$0 = b_*^{(2)}(M') = [M' : M] \cdot b_*^{(2)}(M)$$

and by Lemma 2.16 we have $\tau^{(2)}(M) = \tau^{(2)}(M')^{[M':M]} = 1$.

If N is a complete hyperbolic 3-manifold of finite volume, then it a classical result that all L^2-Betti numbers are zero (see [Lü02, Theorem 1.62]) first proven by Dodziuk [Do79]. Later it was shown by Lott [Lo92] in the closed case and by Lück and Schick ([LS99]) in the case of cusps that $\tau^{(2)}(N) = e^{\frac{1}{6\pi} \cdot \mathrm{Vol}_{\mathbb{H}}(N)}$.

Summarizing everything we obtain the following theorem.

Theorem 2.33. *[Lü02, Theorem 4.1 & 4.3] Let N be a connected compact irreducible 3-manifold with infinite fundamental group and incompressible boundary or $N = S^1 \times D^2$. We have*

$$\begin{aligned} b_0^{(2)}(N) &= 0, \\ b_1^{(2)}(N) &= -\chi(N), \\ b_k^{(2)}(N) &= 0 \quad \text{for all } k \geqslant 2. \end{aligned}$$

Moreover, if $\chi(N) = 0$ i.e. N has empty or toroidal boundary, then the L^2-torsion is well-defined and we have

$$\tau^{(2)}(N) = e^{\frac{1}{6\pi} \cdot \sum_{i=1}^{k} \mathrm{Vol}_{\mathbb{H}}(N_i)},$$

where $N_1, \ldots, N_k$ are the hyperbolic pieces in the JSJ-decomposition.

CHAPTER 3

Vanishing of relative L^2-Betti numbers for taut sutured manifolds

In this chapter we prove one of the main results of this thesis. We follow the preprint of the author [He18].

3.1. Main theorem and applications

Theorem 3.1 (Main theorem)**.** *Let (M, R_+, R_-, γ) be a connected irreducible balanced sutured manifold. Assume that each component of γ and $R_\pm$ is incompressible and $\pi_1(M)$ is infinite, then the following are equivalent*

1. *the manifold (M, R_+, R_-, γ) is taut,*
2. *the L^2-Betti numbers of (M, R_-) are all zero i.e. $H_*^{(2)}(M, R_-) = 0$,*
3. *the map $H_1^{(2)}(R_- \subset M) \to H_1^{(2)}(M)$ is a weak isomorphism.*

Remark 3.2. The same statement holds true if one replaces R_- with R_+ (see Proposition 3.8).

Remark 3.3. One can drop the assumption that M is connected and replace it with the assumption that every connected component has an infinite fundamental group.

By Gabai's theory of sutured manifold decompositions we obtain the following result about Thurston norm minimizing surfaces in an irreducible 3-manifold N with empty or toroidal boundary.

Theorem 3.4. *Let N be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary and infinite fundamental group. If $\Sigma \hookrightarrow N$ a properly embedded decomposition surface, then the following are equivalent*

1. *Σ is Thurston norm minimizing,*
2. *the L^2-Betti numbers of the pair $(N \setminus\setminus \Sigma, \Sigma_-)$ are zero.*

As an application we have the following theorem first proven by Friedl and Lück with different methods [FL18].

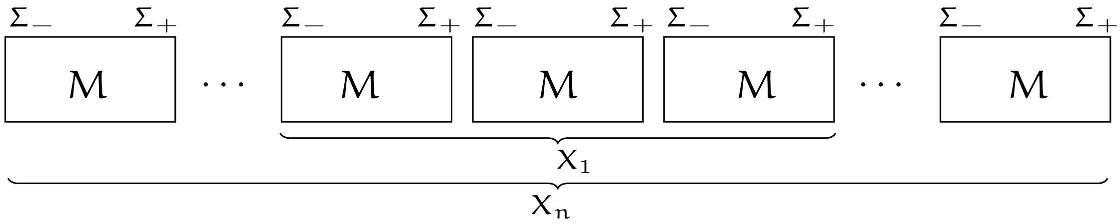

Figure 3.1.: An illustration of the limiting process in the proof of Theorem 3.5.

Theorem 3.5. *Let N be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ be a primitive cohomology class. We write $N_{\ker\phi} \to N$ for the cyclic covering corresponding to $\ker\phi$. We have*

$$b_1^{(2)}(N_{\ker\phi}) = \|\phi\|.$$

Proof . Let $G = \ker\phi$ and Σ be a taut surface Poincaré dual to ϕ. We write $M = N\setminus\!\setminus\Sigma$ and construct inductively

$$X_0 = M,$$
$$X_n = M \bigsqcup_{\Sigma_-=\Sigma_+} X_{n-1} \bigsqcup_{\Sigma_-=\Sigma_+} M.$$

Since ϕ is primitive we have $\lim_{n\in\mathbb{N}} X_n = N_{\ker\phi}$ by Lemma 1.31. We refer to Figure 3.1 for an illustration of the situation.

Also note that all inclusions $\Sigma \to X_1 \to X_n$ and $X_n \to N_\phi$ are π_1-injective by Theorem 1.33. Hence we can use the induction principle (Theorem 2.3 (7)).

By the excision isomorphism we have $(X_n, X_{n-1}) = (M, \Sigma_+) \sqcup (M, \Sigma_-)$ and hence by the main theorem

$$b(X_n, X_{n-1}; \mathcal{N}(G)) = b^{(2)}(X_n, X_{n-1}) = 0.$$

We can consider the triple (X_n, X_{n-1}, Σ) and its associated long exact sequence in homology. It follows inductively that $b(X_n, \Sigma; \mathcal{N}(G)) = 0$.

By Theorem B.1 (5) we also have the isomorphisms

$$\varinjlim_{n\in\mathbb{N}} H_*(X_n, \Sigma; \mathcal{N}(G)) \cong H_*(\varinjlim_{n\in\mathbb{N}} X_n, \Sigma; \mathcal{N}(G)) = H_*(N_{\ker\phi}, \Sigma; \mathcal{N}(G)).$$

Then by cofinality of the von Neumann dimension (Theorem 2.3 (8)) we have

$$0 = \lim_{n\to\infty} b_*(X_n, \Sigma; \mathcal{N}(G)) = b_*^{(2)}(N_{\ker\phi}, \Sigma).$$

We look at the long exact sequence in homology associated to the pair $(N_{\ker\phi}, \Sigma)$ and conclude from the additivity of the von Neumann dimension (see Theorem 2.3 (2)) that

$$b_1^{(2)}(N_\phi) = b_1^{(2)}(\Sigma) = -\chi(\Sigma) = \|\phi\|. \qquad \square$$

Another application of Theorem 3.1 is that for a taut sutured manifold the L^2-torsion is well defined.

Corollary 3.6. *If (M, R_+, R_-, γ) is a taut sutured manifold satisfying the assumptions of Theorem 3.1, then the pair (M, R_-) is L^2-det-acyclic and the L^2-torsion $\tau^{(2)}(M, R_-)$ is well-defined.*

Proof . This follows from Theorem 3.1, Lemma 2.12, and Theorem 2.25. □

3.2. Supplement results

In this section we prove the equivalence of statements (2) and (3) in Theorem 3.1. Moreover, we prove a vanishing criterion for the projective L^2-homology of a cyclic covering of a sutured manifold.

As mentioned in the beginning we state every result only for the pair (M, R_-), but by Poincaré-Lefschetz duality (see Theorem B.13) all results hold equally for the pair (M, R_+). This is the content of the next lemma.

Lemma 3.7. *Let (M, R_+, R_-, γ) be a connected sutured manifold with infinite fundamental group and let γ be incompressible. Then*

$$b_i^{(2)}(M, R_+) = b_i^{(2)}(M, R_+ \cup \gamma)$$

and

$$b_i^{(2)}(M, R_-) = b_{3-i}^{(2)}(M, R_+).$$

Proof . We abbreviate $\pi_1(M) = \pi$ and consider the short exact sequence of chain complexes coming form the Mayer-Vietoris sequence applied to the decomposition $(M, R_+ \cup \gamma) = (M, R_+) \cup (\gamma, \gamma)$:

$$0 \to C_*(\gamma, \partial_+\gamma; \mathcal{N}(\pi)) \to C_*(M, R_+; \mathcal{N}(\pi)) \oplus C_*(\gamma, \gamma; \mathcal{N}(\pi)) \to C_*(M, R_+ \cup \gamma; \mathcal{N}(\pi)).$$

Note that γ is the union of annuli and tori. Hence it follows from Lemma 2.32 that $\dim_{\mathcal{N}(\pi)} H_*(\gamma, \partial\gamma; \mathcal{N}(\pi)) = 0$ and therefore $b_i^{(2)}(M, R_+) = b_i^{(2)}(M, R_+ \cup \gamma)$. By Theorem B.13 and Equation 2.1 we have

$$b_i^{(2)}(M, R_-) = b_{3-i}^{(2)}(M, R_+ \cup \gamma) = b_{3-i}^{(2)}(M, R_+).$$ □

We also use this duality to obtain a general result about L^2-Betti numbers of balanced sutured manifolds.

Proposition 3.8. *If M is a balanced sutured manifold with infinite fundamental group, then $b_1^{(2)}(M, R_-) = b_2^{(2)}(M, R_-)$ and $b_1^{(2)}(M, R_-) = 0$ implies $b_i^{(2)}(M, R_-) = 0$ for all $i \in \mathbb{N}$.*

Proof . As always we write π for $\pi_1(M)$. One has $b_0^{(2)}(M, R_\pm) = 0$, because $\pi_1(M)$ is infinite (Remark 2.14) and by Lemma 3.7 $b_3^{(2)}(M, R_\mp) = 0$, too. Since M is balanced, we have

$$\chi(R_-) = \frac{\chi(R_+ \cup \gamma \cup R_-)}{2} = \frac{\chi(\partial M)}{2} = \chi(M)$$

and hence $\chi(M, R_-) = 0$. By the relation between L^2-Euler characteristic and Euler characteristic (see Equation (2.2)) we obtain

$$\chi(M, R_-; \mathcal{N}(\pi)) = \chi(M, R_-) = 0$$

and thus $b_1^{(2)}(M, R_-) = b_2^{(2)}(M, R_-)$. □

This gives us already the equivalence of (2) and (3) in Theorem 3.1:

Corollary 3.9. *Let (M, γ) be an irreducible balanced sutured manifold. Assume that γ is incompressible and $\pi_1(M)$ is infinite. Then $b_*(M, R_-) = 0$ if and only if $H_1^{(2)}(R_- \subset M) \to H_1^{(2)}(M)$ is a monomorphism.*

Proof . We look at the projective long exact sequence (Lemma 2.5) in L^2-homology of the pair (M, R_-). Note that by our assumptions and Theorem 2.33 we have $b_2^{(2)}(M) = 0$. Then Theorem 2.3(5) implies $\mathbf{P}\, H_2(M; \mathcal{N}(\pi)) = H_2^{(2)}(M) = 0$. Therefore, the sequence becomes

$$0 \longrightarrow H_1^{(2)}(M, R_-) \longrightarrow H_1^{(2)}(R_- \subset M) \longrightarrow H_1^{(2)}(M) \longrightarrow \ldots .$$

Now the corollary follows from Proposition 3.8 and Theorem 2.3 (5). □

Lemma 3.10. *Let (M, γ) be a connected sutured manifold and let $\phi \in H^1(M; \mathbb{Z})$ be nontrivial. If there is a decomposition surface S such that the class $[S] \in H_2(M, \partial M; \mathbb{Z})$ is Poincaré dual to ϕ and $M \overset{S}{\rightsquigarrow} M'$ results in a product sutured manifold $(M', R'_+ . R'_-, \gamma')$, then*

$$\dim_{\mathcal{N}(\mathbb{Z})} H_*^{\phi,(2)}(M, R_-) = 0.$$

It is worth discussing a simple case before we give the proof. Namely, if N is sutured manifold with empty or toroidal boundary where the sutured manifold structure is given by $\gamma = \partial N$. In this case one has $R_- = \emptyset$. Moreover, if S is a decomposition surface S such that the decomposition $N \overset{S}{\rightsquigarrow} N'$ results in a product sutured manifold, then S is a fibre of a fibration of N over S^1. The cover corresponding to $\ker(\phi)$ is homeomorphic to $S \times \mathbb{R}$ and hence one has $H_*(N; \mathbb{C}(t)^\phi) = H_*(S \times \mathbb{R}; \mathbb{Z}) \otimes_{\mathbb{Z}[t^\pm]} \mathbb{C}(t) = 0$.

Proof of Lemma 3.10 . We have two canonical embeddings of S into the boundary of M' which we denote by $i_\pm \colon S \to M'$. Moreover, we denote by $S_\pm$ the images $i_\pm(S)$.

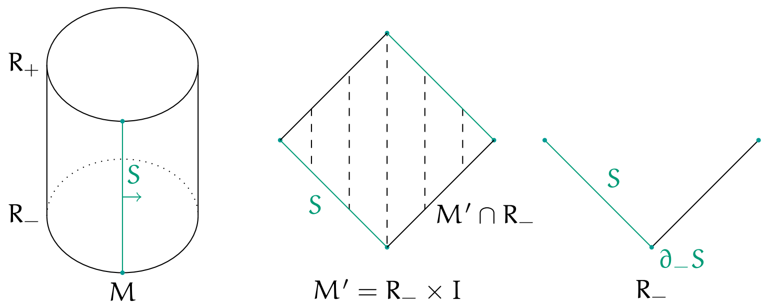

Figure 3.2.: A schematic picture one dimension reduced. The sutured decomposition $M \overset{S}{\rightsquigarrow} M'$ results in a product sutured manifold. The dashed lines show the $[0,1]$-factor of the product. By homotopy invariance and excision one has an isomorphism $H_*(S, \partial_- S) \xrightarrow{i_-} H_*(M', M' \cap R_-)$.

By Remark 2.10 we can assume that ϕ is surjective. Since ϕ is the Poincaré dual of $[S]$, the cyclic cover $p\colon \widehat{M} \to M$ corresponding to $\ker \phi$ can be described by

$$\widehat{M} = (M' \times \mathbb{Z})/\sim,$$

where (S_-, i) is glued to $(S_+, i+1)$ in the obvious way (see Lemma 1.31). The deck transformation group acts on the $\mathbb{Z}$-factor. We denote by t the generator of this action i. e. $t \cdot (x, i) = (x, i+1)$ for all $x \in M'$.

We decompose $\widehat{M}$ into the subsets $\{M' \times \{i\}\}_{i \in \mathbb{Z}}$ and we set $\partial_- S = S \cap R_-$. We write $\widehat{R}_- = p^{-1}(R_-)$. For the rest of the proof we abbreviate $C_*(X) = C_*(X;\mathbb{Z})$ and $H_*(X) = H_*(X;\mathbb{Z})$. We consider the short exact sequence of chain complexes:

$$0 \to \mathbb{Z}[t^{\pm}] \otimes_{\mathbb{Z}} C_*(S, \partial_- S) \xrightarrow{i_- - t i_+} \mathbb{Z}[t^{\pm}] \otimes_{\mathbb{Z}} C_*(M', M' \cap R_-) \to C_*(\widehat{M}, \widehat{R}_-) \to 0. \; .$$

This sequence yields a long exact sequence in homology:

$$\ldots \to \mathbb{Z}[t^{\pm}] \otimes_{\mathbb{Z}} H_k(S, \partial_- S) \xrightarrow{i_- - t i_+} \mathbb{Z}[t^{\pm}] \otimes_{\mathbb{Z}} H_k(M', M' \cap R_-) \to H_k(\widehat{M}, \widehat{R}_-) \to \ldots .$$

Next we show that the inclusion $i_-\colon (S, \partial_- S) \to (M', M' \cap R_-)$ induces an isomorphism on homology.

By assumption M' is a product sutured manifold. This means that $M' = R'_- \times I$ with $R'_- = (M' \cap R_-) \cup S_-$. Therefore, we get by homotopy invariance

$$H_*(M', M' \cap R_-) = H_*(R'_- \times I, M' \cap R_-) \cong H_*(R'_-, M' \cap R_-)$$

and by excision

$$H_*(R'_-, M' \cap R_-) = H_*((M' \cap R_-) \cup S_-, M' \cap R_-) \xleftarrow{i_-} H_*(S, \partial_- S).$$

We refer to Figure 3.2 for an illustration of this argument.

We now continue with the rest of the proof. Because $H_*(M', M' \cap R_-) \cong H_*(S, \partial_- S)$ is free abelian it makes sense to talk about the determinant. And since i_- induces an isomorphism on homology we have $\det_{\mathbb{Z}}(i_-) \neq 0$. This of course implies that $\det_{\mathbb{Z}[t^\pm]}(i_- - t \cdot i_+) \neq 0$. Therefore, the map $i_- - t \cdot i_+$ is invertible over $\mathbb{C}(t)$. Note that $H_*(\widehat{M}, \widehat{R}_-)$ is isomorphic to $H_*(M, R_-; \mathbb{Z}[t^\pm]^\phi)$.

We use the above sequence, the fact that $\mathbb{C}(t)$ is flat over $\mathbb{Z}[t^\pm]$ and that $i_- - t \cdot i_+$ is invertible over $\mathbb{C}(t)$ to obtain $H_*(M, R_-; \mathbb{C}(t)^\phi) = 0$. Now Equation (2.4) yields $H_*^{\phi,(2)}(M, R_-) = 0$. □

3.3. Proof of the main theorem

Vanishing implies taut

The basic idea is that $b_2^{(2)}(M, R_-)$ is an upper bound for how far a sutured manifold (M, γ) is away from being taut. We start with a lemma which shows that we find a taut representative of the class $[R_-] \in H_2(M, \gamma; \mathbb{Z})$ which separates R_+ from R_-.

Lemma 3.11. *If (M, γ) is a connected irreducible sutured manifold with infinite fundamental group and γ is incompressible, then there exists a Thurston norm minimizing representative $S \subset M \setminus (R_+ \cup R_-)$ of $[R_-] \in H_2(M, \gamma; \mathbb{Z})$ such that M cut along S is the union of two disjoint (not necessarily connected) compact manifolds $M_\pm$ with $R_\pm \subset \partial M_\pm$.*

Proof . Let T be a properly embedded surface which is homologous to $[R_-]$, satisfies $\chi_-(T) = \|[R_-]\|_{(M,\gamma)}$ and is disjoint from $R_\pm$. We will show that a subsurface of T has the desired properties.

First we find a subsurface S of T which is homologous to T, has the same complexity, does not have sphere or disc components, and is incompressible.

Since M is irreducible every sphere component of T is null homologous and we can omit it. Now let $D \subset T$ be a disc component. From the definition one has $\partial D \subset \gamma$. Therefore, we have to consider two cases. The first case that ∂D is homotopically trivial in γ. In this case we can close the circle with another disc in γ and obtain a sphere. By the assumption M is irreducible and therefore this sphere bounds a 3-ball. Hence the disc D is homologically trivial in $H_2(M, \gamma; \mathbb{Z})$. Therefore, $[T] = [T \setminus D]$ and we define $S := T \setminus D$. Note that S has the same complexity as T. The other case that ∂D is non trivial in γ can not occur because γ is incompressible. That S is indeed incompressible is a consequence of the loop theorem and the fact that it has minimal complexity. See also [AFW15, Chapter 3 C.22] or [Ca07, Lemma 5.7].

It remains to show that S separates the manifold M into at least two disjoint parts. We have an intersection form:

$$H_2(M, \gamma; \mathbb{Z}) \times H_1(M, R_+ \cup R_-; \mathbb{Z}) \to \mathbb{Z}$$

If p is a path from R_+ to R_-, then the intersection number of $[p]$ and $[R_-]$ is equal to 1. So every surface homologous to $[R_-]$ has to intersect p at least once and therefore separates R_- and R_+. $\square$

Lemma 3.12 (Half lives, half dies). *If W is a compact connected $(2k+1)$-dimensional manifold, then*

$$\dim_{\mathcal{N}(\pi_1(W))} \ker\Big(i_*\colon H_k^{(2)}(\partial W \subset W) \to H_k^{(2)}(W)\Big) = \frac{1}{2} \cdot b_k^{(2)}(\partial W \subset W).$$

Proof . We consider the inclusion induced map $i_*\colon H_k^{(2)}(\partial W \subset W) \to H_k^{(2)}(W)$. We set $\pi = \pi_1(W)$. Because of the additivity of the von Neumann dimension one has

$$b_k^{(2)}(\partial W \subset W) = \dim_{\mathcal{N}(\pi)} \ker(i_*) + \dim_{\mathcal{N}(\pi)} \operatorname{Im}(i_*).$$

Therefore, it is sufficient to prove that $\dim_{\mathcal{N}(\pi)} \ker(i_*) = \dim_{\mathcal{N}(\pi)} \operatorname{Im}(i_*)$. To show this one considers the long exact sequence in homology associated to the pair $(\partial W, W)$. This sequence can be decomposed into long exact sequences

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H_{k+1}^{(2)}(W,\partial W) & \longrightarrow & \ker(i_*) & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H_{k+1}^{(2)}(W,\partial W) & \longrightarrow & H_k^{(2)}(\partial W \subset W) & \longrightarrow & H_k(W) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & \operatorname{Im}(i_*) & \longrightarrow & H_k^{(2)}(W) & \longrightarrow & \dots\ .
\end{array}$$

One has $b_{k+1+i}^{(2)}(W,\partial W) = b_{k-i}^{(2)}(W)$ and $b_{k+i}^{(2)}(\partial W \subset W) = b_{k-i}^{(2)}(\partial W \subset W)$ by Poincaré duality. So we see that the Euler characteristic of the upper and the lower exact sequence contains the same summands except for $\dim_{\mathcal{N}(\pi)} \ker(i_*)$ and $\dim_{\mathcal{N}\pi} \operatorname{Im}(i_*)$. But both sequences have zero Euler characteristic, because they are exact. Hence $\dim_{\mathcal{N}(\pi)} \ker(i_*) = \dim_{\mathcal{N}(\pi)} \operatorname{Im}(i_*)$. $\square$

Lemma 3.13. *With the notation of the Lemma 3.11 and the additional assumption that R_- is incompressible one has*

$$\frac{1}{2}\big(\chi(S) - \chi(R_-)\big) \leqslant b_2^{(2)}(M, R_-).$$

Proof . Applying Mayer-Vietoris on $U := R_- \cup S$ and $V = \gamma'$ for the boundary $\partial M_- = R_+ \cup \gamma' \cup S$ yields an weak isomorphism

$$H_1^{(2)}(\partial M_- \subset M) \cong H_1^{(2)}(R_- \subset M) \oplus H_1^{(2)}(S \subset M).$$

Here we used that $\gamma' \subset \gamma$ is incompressible in M. We set $\pi = \pi_1(M)$ and consider all L^2-homology with the coefficient system coming from M. Therefore, we drop "$\subset M$" from the notation. From Lemma 3.12 applied to the boundary of M_-, we get

$$\frac{1}{2}\big(b_1^{(2)}(R_-) + b_1^{(2)}(S)\big) = \dim_{\mathcal{N}(\pi)} \ker\big(H_1^{(2)}(\partial M_-) \to H_1^{(2)}(M_-)\big).$$

By the standard inequality

$$\dim_{\mathcal{N}(\pi)} \ker(i\colon A \oplus B \to C) \leqslant \dim_{\mathcal{N}(\pi)} \ker(i\colon A \to C) + \dim_{\mathcal{N}G} B$$

applied to $H_1^{(2)}(\partial M_-) = H_1^{(2)}(R_-) \oplus H_1^{(2)}(S) \to H_1(M_-)$ we obtain further

$$\begin{aligned}
\frac{1}{2}\big(b_1^{(2)}(R_-) + b_1^{(2)}(S)\big) &= \dim_{\mathcal{N}(\pi)} \ker\big(H_1^{(2)}(\partial M_-) \to H_1^{(2)}(M_-)\big)\\
&\leqslant \dim_{\mathcal{N}(\pi)} \ker\big(H_1^{(2)}(R_-) \to H_1^{(2)}(M_-)\big) + b_1^{(2)}(S)\\
&\leqslant \dim_{\mathcal{N}(\pi)} \ker\big(H_1^{(2)}(R_-) \to H_1^{(2)}(M)\big) + b_1^{(2)}(S)\\
&\leqslant \dim_{\mathcal{N}(\pi)} \operatorname{Im}\big(H_2^{(2)}(M, R_-) \to H_1^{(2)}(R_-)\big) + b_1^{(2)}(S)\\
&\leqslant b_2^{(2)}(M, R_-) + b_1^{(2)}(S).
\end{aligned}$$

Recall that we dropped "$\subset M$" from the notation but by assumption R_- and S are incompressible and hence $b_1^{(2)}(R_- \subset M) = b_1^{(2)}(R_-)$ and $b_1^{(2)}(S \subset M) = b_1^{(2)}(S)$. For surfaces with infinite fundamental group one has $-\chi(S) = b_1^{(2)}(S)$ and $-\chi(R_-) = b_1^{(2)}(R_-)$, which finishes the proof. □

Now the direction $(2) \Rightarrow (1)$ of the main theorem follows easily.

Corollary 3.14. *Let (M, γ) be an irreducible connected balanced sutured manifold with infinite fundamental group and such that γ and R_- are incompressible. If $b_2^{(2)}(M, R_-) = 0$, then M is taut.*

Proof . Let S be a surface obtained from Lemma 3.11. By construction of S one has

$$-\chi(S) = \|[R_-]\|_{(M,\gamma)}$$

and hence

$$0 \leqslant -\chi(R_-) - \|[R_-]\|_{(M,\gamma)} = -\chi(R_-) + \chi(S).$$

By assumption we have $b_2^{(2)}(M, R_-) = 0$ and Lemma 3.13 implies that

$$0 \leqslant \chi(S) - \chi(R_-) = -\|[R_-]\|_{(M,\gamma)} - \chi(R_-) \leqslant 0.$$

Therefore, we get $\|[R_-]\|_{(M,\gamma)} = -\chi(R_-)$ and since (M, γ) is balanced we obtain $\|[R_-]\|_{(M,\gamma)} = -\chi(R_+)$, too. But this is the definition of a taut sutured manifold. □

Taut implies vanishing

We are now going to show that if (M, γ) is taut, then $H_*^{(2)}(M, R_-)$ is zero. Since L^2-Betti numbers are multiplicative under finite covers (Proposition 2.9) and a finite cover of a taut sutured manifold is again taut (Corollary 1.26) the above statement is true if and only if it is true for a finite cover. The proof consists of three steps:

1. There exists a finite cover $\widehat{M} \to M$ and a decomposition surface $S \subset \widehat{M}$ such that $\widehat{M} \overset{S}{\rightsquigarrow} \widehat{M}'$ is a product sutured manifold.
2. If $\phi \in H^1(\widehat{M}; \mathbb{Z})$ is the Poincaré dual of S, then $H_*^{\phi,(2)}(\widehat{M}, \widehat{R_-}) = 0$.
3. We combine the second step with Schick's approximation result to show that $H^{(2)}(M, R_-) = 0$.

Note that the first step uses the virtual fibring theorem of Agol.

Theorem 3.15. *If M is a connected taut sutured manifold with infinite fundamental group and γ is incompressible, then $b_*^{(2)}(M, R_-) = 0$.*

Proof . Let M be a taut sutured manifold. Recall that for any finite cover $\widehat{M} \to M$ we have

$$[\widehat{M} : M] \cdot b_*^{(2)}(M, R_-) = b_*^{(2)}(\widehat{M}, \widehat{R}_-)$$

and if M is taut, then $\widehat{M}$ is taut as well. Therefore, it is sufficient to prove the theorem for a suitable finite cover. By this observation together with Proposition 1.42 we will assume that (M, γ) admits a decomposition surface S such that M' defined by $M \overset{S}{\rightsquigarrow} M'$ is a product sutured manifold and $[S] \in H_2(M, \partial M; \mathbb{Z})$ is non-trivial. If we denote by $\phi \in H^1(M; \mathbb{Z})$ the Poincaré dual of S, then by Lemma 3.10 we have

$$H_1^{\phi,(2)}(M, R_-) = 0.$$

Since the fundamental group of M is residually finite we obtain a nested sequence $\pi_1(M) = \pi \supset \pi_1 \supset \pi_2 \ldots$ of normal subgroups such that π/π_i is finite and $\bigcap_{i \in I} \pi_i = \{e\}$. Denote by $p_i \colon M_i \to M$ the corresponding finite cover. Denote by $S_i := p_i^{-1}(S)$ the pre-image of the surface S and by $\phi_i = p_i^*(\phi)$ the pull back of ϕ. Obviously, S_i is the Poincaré dual of ϕ_i. Furthermore, $M_i \overset{S_i}{\rightsquigarrow} M_i'$ results in a product sutured manifold since M_i' is a cover of M' and M' is a product sutured manifold. Hence we can apply Lemma 3.10 again to obtain

$$H_1^{\phi_i,(2)}(M_i, R_{i-}) = 0.$$

Denote by $G_i = ker(\phi) \cap \pi_i$ the kernel of ϕ_i. We see that G_i is normal in π. Moreover, by the third isomorphism theorem we have

$$(\pi/G_i)/(\pi_i/G_i) \cong \pi/\pi_i$$

and hence $(\pi_i/G_i) \triangleleft (\pi/G_i)$ is finite index. This yields

$$\begin{aligned} 0 = \dim_{\mathcal{N}(\mathbb{Z})} H_1^{\phi_i,(2)}(M_i, R_i) &= b_1(M_i, R_i; \mathcal{N}(\pi_i/G_i)) \\ &= [\pi : \pi_i] \cdot b_1(M, R; \mathcal{N}(\pi/G_i)) \end{aligned}$$

and in particular

$$b_1(M, R_-; \mathcal{N}(\pi/G_i)) = 0.$$

The groups π/G_i are by construction virtually cyclic and hence lie in the class $\mathcal{G}$. We apply Schick's approximation theorem (Theorem 2.13) to the nested and cofinal sequence of normal subgroups $\pi_1(M) \supset G_1 \supset G_2 \ldots$ and obtain

$$b_1^{(2)}(M, R_-) = b_1(M, R_-; \mathcal{N}(\pi)) = \lim_{i\to\infty} b_1(M, R_-; \mathcal{N}(\pi/G_i)) = 0. \qquad \square$$

CHAPTER 4

Relative torsion

4.1. Basic properties and gluing formulas

Let (M, R_+, R_-, γ) be a taut sutured manifold. If we are in the situation of Theorem 3.1, then by Corollary 3.6 the number $\tau^{(2)}(M, R_-)$ is well-defined. In this chapter we will study this number. For this purpose we abbreviate the assumptions on (M, R_+, R_-, γ) of Theorem 3.1.

Assumption 4.0. A sutured manifold (M, R_+, R_-, γ) satisfies these assumptions if

1. M is balanced,
2. for each connected component of M the fundamental group is infinite,
3. R_+, R_-, and γ are incompressible.

Moreover, if N is a compact oriented irreducible 3-manifold with empty or toroidal boundary and $\phi \in H^1(N; \mathbb{Z})$ a cohomology class, then we can consider a taut surface Σ in N which is Poincaré dual to ϕ. The main goal of this chapter is to prove that the number $\tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_-)$ is an invariant for the pair (N, ϕ).

As a disclaimer, we use versions of Theorem 1.33 together with Lemma 2.27 a lot and we are not pointing out every single usage.

We start with an observation that tori are not visible by L^2-invariants, which reflects the fact that they also have an ambiguous role for taut sutured manifolds.

Lemma 4.1. *Let (M, R_+, R_-, γ) be a taut sutured manifold satisfying Assumption 4.0. If $T \subset \gamma$ is a collection of tori, then $(M, R_+ \cup T, R_-, \gamma \setminus T)$ and $(M, R_+, R_- \cup T, \gamma \setminus T)$ are taut sutured manifolds and one has*

$$\tau^{(2)}(M, R_-) = \tau^{(2)}(M, R_- \cup T).$$

Proof . We look at the short exact sequence of chain complexes

$$0 \to C_*^{cw}(R_- \cup T, R_-; \mathcal{N}(\pi)) \to C_*^{cw}(M, R_-; \mathcal{N}(\pi)) \to C_*^{cw}(M, R_- \cup T; \mathcal{N}(\pi)) \to 0.$$

From the fact that $\tau^{(2)}(T) = 1$ one easily deduces $\tau^{(2)}(R_- \cup T, R_-) = 1$. So Lemma 2.23 applied to the above sequence proves this lemma. $\square$

Lemma 4.2. *Let (M, R_+, R_-, γ) be a taut sutured manifold satisfying Assumption 4.0. Let $(C, \partial C) \to (M, R_+ \cup R_-)$ be a collection of disjointly properly embedded incompressible annuli and tori. If for each annulus component $A \subset C$ the intersections $\partial A \cap R_+$ and $\partial A \cap R_-$ are non-empty, then the sutured manifold decomposition $M \stackrel{C}{\rightsquigarrow} M'$ results in a taut sutured manifold satisfying Assumption 4.0. Moreover, we have*

$$\tau^{(2)}(M, R_-) = \tau^{(2)}(M', R'_-).$$

Proof . With out lost of generality we can assume that M is connected. As usual we abbreviate $\pi = \pi_1(M)$. From Lemma 1.21 one sees that M' is indeed a taut sutured manifold. Moreover, γ' and $R'_\pm$ are still incompressible and hence M' satisfies Assumption 4.0. Note that all components of C are S^1-bundles and hence have trivial L^2-torsion by Lemma 2.32. Therefore, the proof consist of repeatedly applying the gluing formula for L^2-torsion (Lemma 2.29).

Let $T \subset C$ be the union of all torus components of C and $A = C \setminus T$ all annulus components. We have the taut sutured manifold $(M \setminus\setminus T, R_+ \cup T_+, R_- \cup T_-, \gamma)$ coming from the decomposition $M \stackrel{T}{\rightsquigarrow} M \setminus\setminus T$. We first observe that

$$\tau^{(2)}(M \setminus\setminus T, R_- \cup T_-) = \tau^{(2)}(M \setminus\setminus T, R_-)$$

by using Lemma 4.1. By the same arguments one also proves

$$\tau^{(2)}(M, R_-) = \tau^{(2)}(M \setminus\setminus T, R_-) = \tau^{(2)}(M \setminus\setminus T, R_- \cup T_-).$$

Now we can consider the taut sutured manifold decomposition $M \setminus\setminus T \stackrel{A}{\rightsquigarrow} M'$. Let $\nu(A)$ be a tubular neighbourhood of A. By Lemma 2.32 we have $\tau^{(2)}(\nu(A)) = 1$ and $\tau^{(2)}(\nu(\partial A) \cap R_-) = 1$. We can again apply Lemma 2.29, but this time to the decomposition

$$(M \setminus\setminus T, R_- \cup T_-) = (M', R'_-) \cup (\nu(A), \nu(\partial A) \cap R_-))$$

to obtain $\tau^{(2)}(M \setminus\setminus T, R_- \cup T_-) = \tau^{(2)}(M', R'_-)$. Putting everything together finishes the proof of this lemma. □

We obtain a gluing formula which is useful in view of the JSJ-decomposition of a 3-manifold.

Proposition 4.3. *Let N be connected compact oriented irreducible 3-manifold with empty or toroidal boundary and $\mathcal{T}$ a collection of incompressible disjointly embedded tori in N. Let Σ be a non-empty taut surface in N and denote by $M_1, \ldots, M_n$ the connected components of $N \setminus\setminus \mathcal{T}$. We write $\Sigma_i = M_i \cap \Sigma$ and $\Sigma_{i-} = \Sigma_- \cap M_i$. By Proposition 1.9 we can assume and will assume that each Σ_i is Thurston norm minimizing. If Σ_i is a decomposition surface for all $i \in \{1, \ldots, n\}$, then*

$$\tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_-) = \prod_{i=1}^{n} \tau^{(2)}(M_i \setminus\setminus \Sigma_i, \Sigma_{i-}).$$

Proof . We consider $C \coloneqq \mathcal{T} \cap (N \setminus\!\!\setminus \Sigma)$. Our hypothesis about the intersection of Σ_i with the boundary components of $M_1, \dots, M_n$ ensures that C satisfies the assumption of Lemma 4.2, so that this proposition is a direct consequence of Lemma 4.2. □

Another basic property is a Poincaré duality statement for the relative torsion.

Lemma 4.4. *If* (M, R_+, R_-, γ) *is a connected taut sutured manifold satisfying Assumption 4.0, then*

$$\tau^{(2)}(M, R_+) = \tau^{(2)}(M, R_-).$$

Proof . We abbreviate $\pi = \pi_1(M)$. By taking a simplicial structure we can use Theorem B.13 to obtain a chain homotopy equivalence:

$$\frown [M]\colon C^{3-*}_{\mathrm{CW}}(M, R_- \cup \gamma; \mathbb{Z}[\pi]) \to C^{\mathrm{CW}}_*(M, R_+; \mathbb{Z}[\pi]).$$

Lets abbreviate $f = \frown [M]$. From the short exact sequence associated to the cone one obtains

$$\mathrm{cone}_*(f \otimes_{\mathbb{Z}[\pi]} \mathrm{Id}_{\mathcal{N}(\pi)}) = C^{\mathrm{CW}}_*(M, R_+; \mathcal{N}(\pi)) \cdot C^{3-*}_{\mathrm{CW}}(M, R_- \cup \gamma; \mathcal{N}(\pi))^{-1}.$$

We claim that $\tau^{(2)}(\mathrm{cone}_*(f \otimes_{\mathbb{Z}[\pi]} \mathrm{Id}_{\mathcal{N}(\pi)})) = 1$. Note that $\mathrm{cone}_*(f)$ represents an element in the Whitehead group of $\mathbb{Z}[\pi]$. If we represent this element by an $n \times n$-matrix A over $\mathbb{Z}[\pi]$, then as explained in Lück's book [Lü02, Section 3] one has $\tau^{(2)}(\mathrm{cone}_*(f \otimes_{\mathbb{Z}[\pi]} \mathrm{Id}_{\mathcal{N}(\pi)})) = \det_{\mathcal{N}(\pi)} A$. In particular, the L^2-torsion is independent of the choice of representative of A. By a theorem of Milnor [Mi62b, Theorem 1] the element $\mathrm{cone}_*(f)$ in the Whitehead group is trivial and therefore represented by the identity. Hence we have $\tau^{(2)}(\mathrm{cone}_*(f \otimes_{\mathbb{Z}[\pi]} \mathrm{Id}_{\mathcal{N}(\pi)})) = \det_{\mathcal{N}(G)} \mathrm{Id} = 1$ which finishes the proof of the claim.

It follows from the definitions that

$$\tau^{(2)}(C^{3-*}(M, R_- \cup \gamma; \mathcal{N}(\pi))) = \tau^{(2)}(C_*(M, R_- \cup \gamma; \mathcal{N}(\pi))).$$

By Lemma 4.1 one has $\tau^{(2)}(M, R_- \cup \gamma) = \tau^{(2)}(M, R_-)$, which finishes the proof. □

Next we consider some gluing formula for the relative torsion. We refer to Figure 4.1 for an illustration.

Lemma 4.5. *Let* (N_1, S_+, S_-, γ) *and* (N_2, T_+, T_-, γ') *be two disjoint taut sutured manifolds satisfying Assumption 4.0. Let* T'_- *be a union of connected components of* T_-*. Suppose that* T'_- *is homeomorphic to* S'_+*, where* S'_+ *is a union of connected components of* S_+*. We obtain a new taut sutured manifold* (M, R_+, R_-, ν) *with*

$$\begin{aligned} M &= N_1 \sqcup_{S'_+ = T'_-} N_2, \\ R_+ &= (S_+ \setminus S'_+) \cup T_+, \\ R_- &= S_- \cup (T_- \setminus T'_-), \\ \nu &= \gamma \cup \gamma'. \end{aligned}$$

Moreover, we have

$$\tau^{(2)}(M, R_-) = \tau^{(2)}(N_1, S_-) \cdot \tau^{(2)}(N_2, T_-) = \tau^{(2)}(M \setminus\!\!\setminus S'_+, S_- \cup T_-).$$

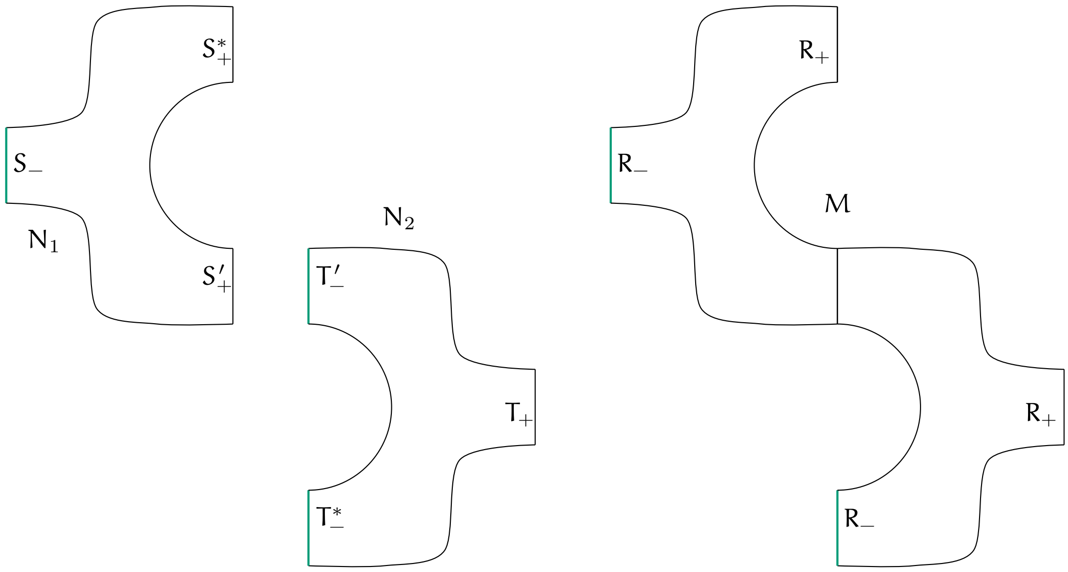

Figure 4.1.: A schematic illustration of Lemma 4.5. It shows the kind of gluing between sutured manifold, which is allowed and does not change the relative torsion. The green part indicates the relative part.

Proof . Without loss of generality, we assume that M is connected. We abbreviate for the proof $\pi = \pi_1(M)$ and $T_-^* = T_- \setminus T_-'$. By our assumptions we have $b_*^{(2)}(N_1, S_-) = b_*^{(2)}(N_2, T_-) = 0$. The triple $R_- \subset N_1 \cup T_-^* \subset M$ induces a short exact sequence of chain complexes:

$$0 \to C_*^{cw}(N_1 \cup T_-^*, R_-; \mathcal{N}(\pi)) \to C_*^{cw}(M, R_-; \mathcal{N}(\pi)) \to C_*^{cw}(M, N_1 \cup T_-^*; \mathcal{N}(\pi)) \to 0.$$

By cellular excision we have

$$C_*^{cw}(M, N_1 \cup T_-^*; \mathcal{N}(\pi)) = C_*^{cw}(N_2, T_-' \cup T_-^*; \mathcal{N}(\pi)) = C_*^{cw}(N_2, T_-; \mathcal{N}(\pi))$$

and

$$C_*^{cw}(N_1 \cup T_-^*, R_-; \mathcal{N}(\pi)) = C_*^{cw}(N_1 \cup T_-^*, S_- \cup T_-^*; \mathcal{N}(\pi)) = C_*^{cw}(N_1, S_-; \mathcal{N}(\pi)).$$

Then Lemma 2.23 applied to the above sequence together with both equalities yields

$$\tau^{(2)}(M, R_-) = \tau^{(2)}(N_1, S_-; \mathcal{N}(\pi)) \cdot \tau^{(2)}(N_2, T_-; \mathcal{N}(\pi)).$$

Note that by our hypothesis the inclusion induced maps $\pi_1(N_i) \to \pi_1(M)$ are monomorphisms. Thus by the induction principle for torsion we have

$$\tau^{(2)}(N_1, S_-; \mathcal{N}(\pi))) = \tau^{(2)}(N_1, S_-) \text{ and } \tau^{(2)}(N_2, T_-; \mathcal{N}(\pi)) = \tau^{(2)}(N_2, T_-).$$

Moreover, we have the sutured manifold decomposition $M \overset{S_+'}{\rightsquigarrow} N_1 \sqcup N_2$ which results in a taut sutured manifold. Since the torsion of a not-connected space is the product of the torsions of the connected components we obtain:

$$\tau^{(2)}(N_1, S_-) \cdot \tau^{(2)}(N_2, T_-) = \tau^{(2)}(M \setminus\setminus S_+', S_- \cup T_-). \qquad \square$$

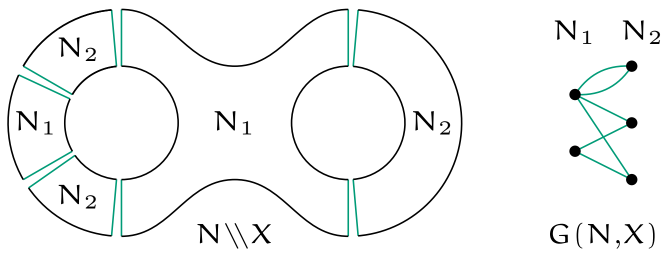

Figure 4.2.: The bipartite structure of the graph $G(N, X)$ decomposes $N \backslash\backslash X$ into two (not necessarily connected) manifolds N_1 and N_2.

4.2. On the combinatorics of Thurston norm minimizing surfaces representing the same class

We use the last section to show that the relative torsion $\tau^{(2)}(N \backslash\backslash \Sigma, \Sigma_-)$ is in fact an invariant of the cohomology class $\phi = D_N([\Sigma])$. In order to succeed we have to study the relation of two Thurston norm minimizing surfaces representing the same homology class.

One of the main tools in this section is Definition 1.32 which translate such question into combinatorics.

The next lemma was communicated to me by José Pedro Quintanilha.

Lemma 4.6. *Let N be a compact oriented 3-manifold and $\phi \in H^1(N; \mathbb{Z})$ be a non-zero cohomology class. If the properly embedded surface X is Poincaré dual to the class $2 \cdot \phi$, then the graph $G(N, X)$ is bipartite.*

Proof . We use the well-known fact, that a graph is bipartite if and only if every circle has even length. A circle $c = (X_1, \ldots, X_n)$ in $G(N, X)$ correspond to a sequence of connected components of X. We can embed a loop γ_c in N intersecting the components $X_1, \ldots X_n$ transversally and otherwise lie in the interior of $N \backslash\backslash X$. Then the length of the circle c is equal to $\#(X \cap \gamma_c) = 2 \cdot \phi(\gamma_c)$. Hence the circle has even length. □

Remark 4.7. The bipartite structure of the graph $G(N, X)$ canonically decomposes $N \backslash\backslash X$ into two manifolds N_1 and N_2. We refer to Figure 4.2 for an illustration.

We can use the above lemma to relate any two Thurston norm minimizing surfaces by a sequence of disjoint Thurston norm minimizing surfaces:

Lemma 4.8. *Let N be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary and $\phi \in H^1(N; \mathbb{Z})$. Let $S, T \to N$ be two embedded surfaces which are Thurston norm minimizing and $[S] = [T] = D_N(\phi) \in H_2(N, \partial N; \mathbb{Z})$. There is a sequence of Thurston norm minimizing surfaces $S_0, \ldots S_n$ with $S = S_0$, $T = S_n$, $S_i \cap S_{i+1} = \emptyset$, and $[S_i] = D_N(\phi) \in H_2(N, \partial N; \mathbb{Z})$ for all $i \in \{1, \ldots n\}$. Moreover, if T and S are taut, then all S_i can be chosen to be taut.*

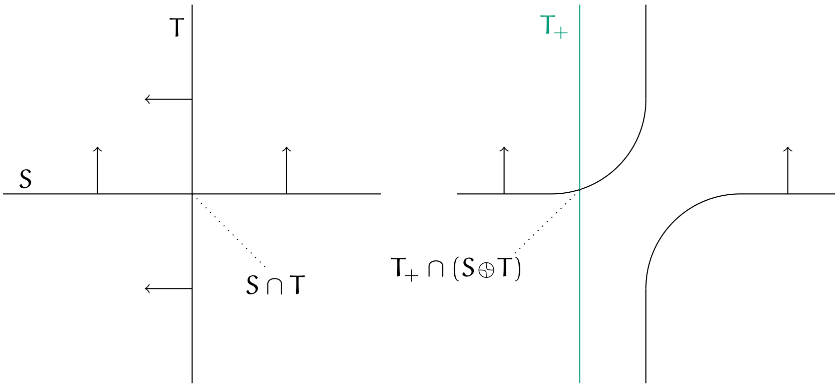

Figure 4.3.: This picture shows the equality of $|\pi_0(S \cap T)| = |\pi_0(T_+ \cap (S \oplus T))|$.

Remark 4.9. This lemma can be seen as a generalisation of a theorem by Kakimizu [Ka92, Theorem A]. He proved this in the special case that N is a link exterior.

Proof . We isotope S and T such that they are in general position and the number of connected components of $S \cap T$ is minimal. The proof is on induction of $|\pi_0(S \cap T)|$. If $|\pi_0(S \cap T)| = 0$, then S and T are disjoint and their is nothing to prove.

Now let $|\pi_0(S \cap T)| > 0$. We consider the surface $S \oplus T$. We have in homology $[S \oplus T] = [S] + [T] = 2 \cdot [T] \in H_2(N, \partial N; \mathbb{Z})$. Hence by Lemma 4.6 the graph $G(N, S \oplus T)$ is bipartite. Using the bipartite structure we can decompose $N \backslash\backslash S \oplus T$ canonically in two manifolds N_1 and N_2 as explained in Remark 4.7.

Let C be the union of connected components of $S \oplus T$ corresponding to all edges going out of the vertices corresponding to N_1 and $D = S \oplus T \setminus C$ the compliment i. e. the union of all connected component corresponding to the edges going into the vertices corresponding to N_1. Then the boundaries of N_1 and N_2 are given by

$$\partial N_1 = C_+ \cup D_- \cup (N_1 \cap \partial N) \text{ and } \partial N_2 = C_- \cup D_+ \cup (N_2 \cap \partial N).$$

From the equation $0 = [\partial N_1] = [C_+] - [D_-] \in H_2(N, \partial N; \mathbb{Z})$ we deduce that [C] is homologous to [D] in $H_2(N, \partial N; \mathbb{Z})$, since C_+ and D_- are isotopic to C and D in N. We further conclude

$$2 \cdot [C] = [C] + [D] = [S \oplus T] = 2 \cdot [S] = 2 \cdot [T].$$

Since $H_2(N, \partial N; \mathbb{Z})$ is torsion free, we see that $[C] = [S] = [T]$. We can conclude from the induction hypothesis once we showed that $|\pi_0(C \cap T)| < |\pi_0(S \cap T)|$, where $\pi_0(C \cap T)$ is understood to be minimal among all isotopy classes of T and the same for S.

One should notice that $|\pi_0((S \oplus T) \cap T_+)| = |\pi_0(S \cap T)|$ (see Figure 4.3). Moreover, we have that C and D are non-empty and $C \cup D = S \oplus T$. Therefore, at least one inequality $|\pi_0(C \cap T_+)| < |\pi_0(S \cap T)|$ or $|\pi_0(C \cap T_-)| < |\pi_0(S \cap T)|$ holds. The same is true for S replaced by T. So we can use the induction hypothesis to obtain a sequence

$$S = S_1, \ldots, S_k, C, S_{k+1}, \ldots S_m = T,$$

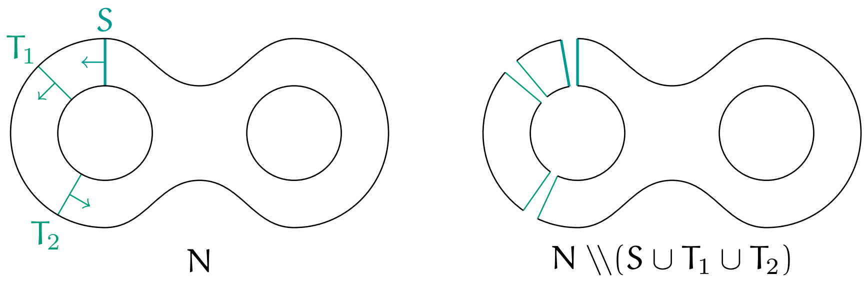

Figure 4.4.: In this picture T_1 is a push-off of S of order 0 and T_2 is a push-off of S of order 1.

where all the S_i's have the desired properties. This shows the first part of the statement. The moreover part of this lemma is a direct consequence of the construction above because the S_i's are subsurfaces of taut surfaces. $\square$

Lemma 4.10. *Let N be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary. Let S and T be two non-empty taut surfaces with $S \cap T = \emptyset$ and $S \cup T$ taut. We further assume that $[S] = [T] \in H_2(N, \partial N; \mathbb{Z})$. Then we have*

$$\tau^{(2)}(N \backslash\backslash S, S_-) = \tau^{(2)}(N \backslash\backslash S \cup T, S_- \cup T_-) = \tau^{(2)}(N \backslash\backslash T, T_-).$$

Proof . Since the role of S and T are symmetric it is sufficient to prove only the equality $\tau^{(2)}(N \backslash\backslash S, S_-) = \tau^{(2)}(N \backslash\backslash S \cup T, S_- \cup T_-)$. The proof will be on induction, but first we have to define a good complexity. Morally the complexity is the number of connected components of T which are not isotopic to a component of S, but it is worth to make this precise.

Let $T_1, \ldots, T_n$ be the connected components of T and $S_1, \ldots S_m$ be the connected components of S. By assumption $N \backslash\backslash(S \cup T)$ is a taut sutured manifold.

We call a connected component T_i of T a push-off of S of order 0 if there is a connected component S_j of S and a product sutured manifold component $M \subset N \backslash\backslash T \cup S$ such that T_i is the plus boundary and S_j the minus boundary of M or vice versa.

We extend the definition of push-off inductively and call T_k a push-off of S of order n if there is a push-off T_i of order $n-1$ and a product sutured manifold component $M \subset N \backslash\backslash T \cup S$ such that T_i is the plus boundary and T_k the minus boundary or vice versa. We refer to Figure 4.4 for an illustration.

If T_k is a push-off of some order, then we simply call it a push-off of S. The complexity that we consider is the number of connected components of T which are *not* push-offs.

For the induction beginning we assume that all surfaces of T are push-offs of S. Hence one has

$$N \backslash\backslash(S \cup T) = N \backslash\backslash S \quad \sqcup \quad T \times [-1, 1]$$

as sutured manifolds. Since the relative torsion of product pieces is trivial (see Theorem 2.25), we have

$$\tau^{(2)}(N \setminus\!\!\setminus S \cup T, S_- \cup T_-) = \tau^{(2)}(N \setminus\!\!\setminus S, S_-).$$

This proves the induction beginning and we continue with the induction step.

We consider the graph $G(N, S \cup T)$. This graph is bipartite by Lemma 4.6. Let N_1 and N_2 be the 3-manifolds corresponding to the bipartite structure (see Remark 4.7). Both are taut sutured manifolds. In order to describe their sutured structure we introduce the following notation.

$$\overset{N_i \to}{S} := \text{Components of } S \text{ corresponding to the edges going out of } N_i.$$
$$\overset{N_i \leftarrow}{S} := \text{Components of } S \text{ corresponding to the edges going into } N_i.$$
$$\overset{N_i \to}{T} := \text{Components of } T \text{ corresponding to the edges going out of } N_i.$$
$$\overset{N_i \leftarrow}{T} := \text{Components of } T \text{ corresponding to the edges going into } N_i.$$

By the nature of a bipartite graph we have equations of the type $\overset{N_1 \to}{S} = \overset{N_2 \leftarrow}{S}$ and thus $S = \overset{N_1 \to}{S} \cup \overset{N_1 \leftarrow}{S}$ and $T = \overset{N_1 \to}{T} \cup \overset{N_1 \leftarrow}{T}$.

If we abbreviate $C = \overset{N_1 \to}{T} \cup \overset{N_1 \to}{S}$ and $D = \overset{N_1 \leftarrow}{T} \cup \overset{N_1 \leftarrow}{S}$, then the sutured manifold structure is described by

$$(N_1, C_+, D_-, N_1 \cap \partial N) \quad \text{and} \quad (N_2, D_+, C_-, N_2 \cap \partial N).$$

Note that $C \cup D = S \cup T$ and since $[C_-] = [D_+] \in H_2(N, \partial N; \mathbb{Z})$ we obtain in homology

$$2 \cdot [C_-] = 2 \cdot [D_+] = [C \cup D] = [S \cup T] = 2 \cdot [S] \in H_2(N, \partial N; \mathbb{Z}).$$

We further get $[C] = [D] = [S]$, because $H_2(N, \partial N; \mathbb{Z})$ is torsion-free.

By assumption there is a component T_j of T such that T_j is not a push-off of S. This component has to be either in $\overset{N_1 \to}{T}$ or $\overset{N_1 \leftarrow}{T}$. If we suppose that $T_j \in \overset{N_1 \leftarrow}{T}$, then we claim that C_- fulfils the induction hypothesis. (In the other case one would work with D_+). In order to see this one recalls the definition of C_-:

$$C_- = \underbrace{\overset{N_1 \to}{T_-}}_{\substack{\text{Components of } T \\ \text{without } T_j}} \cup \underbrace{\overset{N_1 \to}{S_-}}_{\text{push-offs of } S}.$$

Therefore, C_- has strictly fewer push-offs components of S than T has. Hence by induction hypothesis we get

$$\tau^{(2)}(N \setminus\!\!\setminus S, S_-) = \tau^{(2)}(N \setminus\!\!\setminus (C_- \cup S), C_- \cup S). \tag{4.1}$$

Next we consider the taut sutured manifold M obtained from N_1 and N_2 by gluing $\overset{N_1\leftarrow}{T_-}$ to $\overset{N_1\leftarrow}{T_+}$. One gets the equalities:

$$M = N_1 \bigsqcup_{\overset{N_1\leftarrow}{T_-} = \overset{N_1\leftarrow}{T_+}} N_2 = N \backslash\backslash (S \cup (T \setminus \overset{N_1\leftarrow}{T})) = N \backslash\backslash (S \cup \overset{N_1\rightarrow}{T}).$$

Moreover, one sees that

$$\begin{aligned} N \backslash\backslash (C_- \cup S) &= N \backslash\backslash (\overset{N_1\rightarrow}{T_-} \cup \overset{N_1\rightarrow}{S_-} \cup \overset{N_1\rightarrow}{S} \cup \overset{N_1\leftarrow}{S}) \\ &= \big(N \backslash\backslash (S \cup \overset{N_1\rightarrow}{T_-})\big) \backslash\backslash \overset{N_1\rightarrow}{S_-} = M \sqcup \overset{N_1\rightarrow}{S} \times [-1, 1]. \end{aligned}$$

We use again the fact that product pieces do not contribute to the torsion. Hence we obtain

$$\tau^{(2)}(N \backslash\backslash (C_- \cup S), C_- \cup S) = \tau^{(2)}(M, S_- \cup \overset{M_1\rightarrow}{T_-}). \tag{4.2}$$

We further see from the gluing formula (Lemma 4.5) and Equation 4.2:

$$\begin{aligned} \tau^{(2)}(N \backslash\backslash (S \cup T), S_- \cup T_-) &= \tau^{(2)}(N_1, D_-) \cdot \tau^{(2)}(N_2, C_-) \\ &= \tau^{(2)}(M, S_- \cup \overset{M_1\rightarrow}{T_-}) \\ &= \tau^{(2)}(N \backslash\backslash (C_- \cup S), C_- \cup S_-). \end{aligned}$$

Together with Equation 4.1 we have

$$\tau^{(2)}(N \backslash\backslash S, S_-) = \tau^{(2)}(N \backslash\backslash (C_- \cup S), C_- \cup S_-) = \tau^{(2)}(N \backslash\backslash (S \cup T), S_- \cup T_-).$$

As mentioned in the beginning of the proof, the complete statement follows from the previous argument by changing the roles of S and T. □

Putting everything together we obtain the following corollary.

Corollary 4.11. *Let* N *be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary. If* S *and* T *are two non-empty taut surfaces with* $[S] = [T] \in H_2(N, \partial N; \mathbb{Z})$, *then have*

$$\tau^{(2)}(N \backslash\backslash S, S_-) = \tau^{(2)}(N \backslash\backslash T, T_-).$$

Proof . Lemma 4.8 states that any two taut surfaces representing the same homology class are related by disjoint taut surfaces all representing the same class. So it is sufficient to prove this corollary for the case that T and S are disjoint, but this is the content of Lemma 4.10. □

For later purpose we state a proposition which is a direct consequence of Corollary 4.11 and Proposition 1.34.

Proposition 4.12. *If Σ is a taut surface in a connected compact oriented irreducible 3-manifold N with empty or toroidal boundary, then there exist a subsurface S' consisting of connected components of Σ such that $\tau^{(2)}(N \setminus\setminus S', S'_-) = \tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_-)$ and $N \setminus\setminus S'$ is connected. Moreover, if $S_1, \ldots, S_n$ are the connected components of S', then there are positive numbers $w_1, \ldots, w_n$ such that $[\Sigma] = \sum_{i=1}^n w_i \cdot [S_i] \in H_2(N, \partial N; \mathbb{Z})$ holds.*

We end this chapter by showing that for the definition of relative torsion we do not need a taut surface. A Thurston norm minimizing surface is good enough. This will be done by a doubling argument.

Proposition 4.13. *Let S be a Thurston norm minimizing surface in a connected compact oriented irreducible 3-manifold $N \neq S^1 \times D^2$ with empty or toroidal boundary. We have $b_*^{(2)}(N \setminus\setminus S, S_-) = 0$ and hence $\tau^{(2)}(N \setminus\setminus S, S_-)$ is well-defined. Moreover, if $D(N)$ is the double of N along the boundary, then we have*

$$\tau^{(2)}(N \setminus\setminus S, S_-)^2 = \tau^{(2)}(D(N) \setminus\setminus D(S), D(S)_-).$$

Proof . We abbreviate $\pi = \pi_1(D(N))$ and consider the short exact sequence of chain complexes:

$$0 \to C_*^{CW}(\partial N, \partial S; \mathcal{N}(\pi)) \to \begin{array}{c} C_*^{CW}(N_1 \setminus\setminus S_{1-}, S_{1-}; \mathcal{N}(\pi)) \\ \oplus \\ C_*^{CW}(N_2 \setminus\setminus S_{2-}, S_{2-}; \mathcal{N}(\pi)) \end{array} \to C_*^{CW}(D(N) \setminus\setminus D(S), D(S)_-; \mathcal{N}(\pi)) \to 0$$

Note that ∂N and ∂S consist of tori and circle. They are incompressible in N by Lemma 1.10. Hence we have $b_*(\partial N, \partial S; \mathcal{N}(\pi)) = 0$. The surface $D(S)$ is Thurston norm minimizing by Lemma 1.11. Because $D(N)$ is closed, $D(S)$ is a decomposition surface and hence by Theorem 3.4 we have $b_*(D(N) \setminus\setminus D(S), D(S)_-; \mathcal{N}(\pi)) = 0$. This implies $b_*^{(2)}(N \setminus\setminus S, S_-; \mathcal{N}(\pi)) = 0$. The rest of the theorem follows from Lemma 2.23. $\square$

This let us relax the assumptions of Proposition 4.3, Corollary 4.11, and Proposition 4.12. However, we do not make use of it and leave it as an exercise.

CHAPTER 5

An upper bound on the leading coefficient

In this chapter we compare the relative torsion from the last chapter to another invariant called the leading coefficient introduced by Liu [Li17].

This is a rewrite of a preprint from the author of this thesis together with Friedl and Ben-Aribi [BFH18]. While the presentation has be changed, the logic is completely analogue to the preprint.

5.1. L^2-Alexander torsion

Let N be a connected compact irreducible 3-manifold. We write $\pi = \pi_1(N)$. Given an element $\phi \in H^1(N;\mathbb{R}) = \mathrm{Hom}_{\mathbb{Z}}(H_1(N),\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(\pi,\mathbb{R})$ and a positive real number $t \in \mathbb{R}$, we can view $\mathcal{N}(\pi)$ as a twisted $\mathbb{Z}[\pi]$-module by letting an element $g \in \pi$ act by $g * A := t^{\phi(g)} g \cdot A$. The resulting $\mathbb{Z}[\pi]$-module is denoted by $\mathcal{N}(\pi)^{(\phi,t)}$. If we pick a CW-structure of N and lifts of cells to the universal cover, then we can consider the function

$$\begin{aligned}\tau^{(2)}(N,\phi)\colon \mathbb{R}_{>0} &\longrightarrow \mathbb{R}_{\geqslant 0}\\ t &\longmapsto \tau^{(2)}(C_*^{\mathrm{CW}}(N,\mathcal{N}(\pi)^{(\phi,t)}).\end{aligned}$$

Definition 5.1. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be two functions. We say that f and g are equivalent if there exists a $r \in \mathbb{R}$ such that $f(t) = t^r g(t)$ holds. If f and g are equivalent, then we write $f \doteq g$

We have the following result about the dependence of choice of lifts and CW-structure.

Proposition 5.2. *[DFL16, Lemma 4.1] The equivalence class of the function $\tau^{(2)}(N,\phi)$ is well-defined.*

This invariant first appeared in a article by Li and Zhang [LZ06]. We call the equivalence class of $\tau^{(2)}(N,\phi)$ the L^2-Alexander torsion. Here we summarise some basic properties of the L^2-Alexander torsion. For more details and proofs we refer to the work of Dubois, Friedl and Lück [DFL16].

Theorem 5.3. *Let N be a compact oriented irreducible 3-manifold with empty or toroidal boundary and $\phi \in H^1(N;\mathbb{R})$ be non-zero. The following statements hold:*

1. *The evaluation of* $\tau^{(2)}(N,\phi)$ *at* $t = 1$ *equals* $e^{\mathrm{Vol}_{\mathbb{H}}(N)/6\pi}$*, where the volume* $\mathrm{Vol}_{\mathbb{H}}(N)$ *of* N *is defined as the sum of the volumes of the hyperbolic pieces in the JSJ-decomposition of* N.

2. *If* $N = N_1 \sqcup N_2$ *is the disjoint union of two 3-manifolds, then*
$$\tau^{(2)}(N,\phi) = \tau^{(2)}(N_1,\phi|_{N_1}) \cdot \tau^{(2)}(N_2,\phi|_{N_2}).$$

3. *Let* M *be compact oriented irreducible 3-manifold with empty or toroidal boundary and let* $M_1, \ldots M_n$ *be the connected components of* M*. Suppose the boundary of* M *is incompressible and* N *is obtained from* M *by gluing some boundary components. For any non-zero cohomology class* $\phi \in H^1(N,\mathbb{Z})$ *one has*
$$\tau^{(2)}(N,\phi) = \prod_{k=1}^{n} \tau^{(2)}(M_k,\phi|_{M_k}).$$

4. *If* ϕ *is integral, i.e.* $\phi \in H^1(N;\mathbb{Z})$*, then the* L^2*-Alexander torsion is symmetric in the sense that*
$$\tau^{(2)}(N,\phi)(t) \;=\; t^k \cdot \tau^{(2)}(N,\phi)(t^{-1}),$$
for some $k \in \mathbb{Z}$.

5. *If* ϕ *is an integral fibred class, then there exists a* $T \geqslant 1$ *such that*
$$\tau^{(2)}(N,\phi)(t) \;\doteq\; \begin{cases} t^{\|\phi\|}, & \textit{if } t > T, \\ 1, & \textit{if } t < \frac{1}{T}. \end{cases}$$
In fact one can take T *to be the entropy of the monodromy of the fibration.*

One problem is that the L^2-torsion is notoriously difficult to compute. The author calculated the L^2-Alexander torsion in his master thesis for graph manifolds [He17].

Theorem 5.4. *If* N *is a graph manifold, then for any* $\phi \in H^1(N,\mathbb{R})$ *one has*
$$\tau^{(2)}(N,\phi)(t) \doteq \max\left\{1, t^{\|\phi\|}\right\}.$$

The L^2-Alexander torsion is morally an L^2-version of the classical Alexander polynomial. Since the degree of the classical Alexander polynomial is an useful quantity, we consider an analogue for the L^2-Alexander torsion.

Definition 5.5. Given a function $\tau\colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, we assume that there are $d, D \in \mathbb{R}$ and non-zero real numbers c, C with the property that
$$\lim_{t\to\infty} \frac{\tau(t)}{t^D} = C \quad \text{and} \quad \lim_{t\to 0^+} \frac{\tau(t)}{t^d} = c.$$
In this case we define the degree of τ to be
$$\deg(\tau) = D - d.$$

Example 5.6. Given a Laurent polynomial $p(t) = \sum_{i=-n}^{m} a_i t^i$ with $a_m, a_{-n} \neq 0$, the degree equals $m + n$.

One easily verifies that for equivalent functions $f \doteq \tau$ one has $\deg(f) = \deg(\tau)$.

If we look at Theorem 5.4, then we see that for a graph manifold N the degree of $\tau^{(2)}(N, \phi)$ is equal to the Thurston norm of ϕ. Note that Theorem 5.3 (5) has a similar conclusion for a fibred class. It is shown by Friedl and Lück in the rational case [FL19] and in the general case independently by Liu [Li17] that the degree of the L^2-Alexander torsion always detects the Thurston norm. The results of Liu are discussed in the next section in more detail.

5.2. The work of Liu and the definition of the leading coefficient

Here we summarise the results of Liu about the L^2-Alexander torsion [Li17]:

Theorem 5.7. *Let $N \neq S^1 \times D^2$ be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary and let $\phi \in H^1(N; \mathbb{R})$ be non-zero. The following statements hold:*

1. *The L^2-Alexander torsion $\tau^{(2)}(N, \phi)\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geqslant 0}$ takes values in $\mathbb{R}_{>0}$.*
2. *The L^2-Alexander torsion $\tau^{(2)}(N, \phi)\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is continuous.*
3. *For any representative τ of $\tau^{(2)}(N, \phi)$ we have*
$$\deg\left(\tau^{(2)}(N, \phi)\right) = \|\phi\|.$$
In particular, both limits in the definition of degree exists.
4. *There exists a constant $C(N, \phi) \in \mathbb{R}_{>0}$ such that for any representative τ of $\tau^{(2)}(N, \phi)$ there exists a $D \in \mathbb{R}$ with*
$$\lim_{t \to \infty} \frac{\tau(t)}{t^D} = C(N, \phi).$$
We refer to $C(N, \phi)$ as the leading coefficient of $\tau^{(2)}(N, \phi)$. This invariant has the following two properties:
   a) *The leading coefficient $C(N, \phi)$ lies in the interval $\left[1, e^{\mathrm{Vol}_{\mathbb{H}}(N)/6\pi}\right]$.*
   b) *The function $H^1(N; \mathbb{R}) \to \mathbb{R}$ given by $\phi \mapsto C(N, \phi)$ is upper semi-continuous.*

By the work of Liu one can study a new invariant for the pair (N, ϕ) namely the leading coefficient of Theorem 5.7 (4). In this thesis we are mostly interested in the case that ϕ is an integral class i. e. $\phi \in H^1(N, \mathbb{Z})$.

From the definition of the leading coefficient and some properties of the L^2-Alexander function we directly derive some basic properties for the leading coefficient.

Lemma 5.8. *Let $N \neq S^1 \times D^2$ be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary and let $\phi \in H^1(N;\mathbb{Z})$ be non-zero. There is a unique representative $\tau(t)$ of the L^2-Alexander torsion $\tau^{(2)}(N,\phi)$ such that*

$$\lim_{t\to 0^+} \tau(t) = C(N,\phi)$$

and

$$\lim_{t\to\infty} \frac{\tau(t)}{t^{\|\phi\|}} = C(N,\phi)$$

holds.

Proof . Let $f(t)$ be a representative of $\tau^{(2)}(N,\phi)$. By Theorem 5.7 (3) we have that both limits

$$\lim_{t\to\infty} \frac{f(t)}{t^D} = C \quad \text{and} \quad \lim_{t\to 0^+} \frac{f(t)}{t^d} = c$$

exists and $D - d = \|\phi\|$. One easily sees that $C = C(N,\phi)$. We basically claim that $c = C$. By Theorem 5.3 (4) we obtain a $k \in \mathbb{Z}$ such that $t^k \cdot f(t) = f(t^{-1})$. Then we conclude

$$C(N,\phi) = \lim_{t\to\infty} \frac{f(t)}{t^D} = \lim_{t\to 0^+} \frac{f(t^{-1})}{t^{-D}} = \lim_{t\to 0^+} \frac{f(t^{-1})}{t^{-D}} = \lim_{t\to 0^+} \frac{f(t)}{t^{-D-k}} = c.$$

This also implies that $d = -D - k$. If we define $\tau(t) := f(t)\cdot t^d$, then $\tau(t)$ has the desired properties. $\square$

Lemma 5.9. *Let M be compact irreducible 3-manifold with empty or toroidal boundary and let $M_1, \dots M_n$ be the connected components of M. Suppose the boundary of M is incompressible and N is an oriented connected 3-manifold obtained from M by gluing some boundary components. We denote by $i_k \colon M_k \to N$ the resulting inclusions. For any non-zero cohomology class $\phi \in H^1(N,\mathbb{Z})$ one has*

$$C(N,\phi) = \prod_{k=1}^{n} C(M_k, i_k^*(\phi)) = \prod_{k=1}^{n} C(M_k, \phi|_{M_k}).$$

Proof . This follows from Theorem 5.3 (3). $\square$

Lemma 5.10. *If N is a compact oriented irreducible 3-manifold with empty or toroidal boundary and $\phi \in H^1(N;\mathbb{Z})$ non-zero, then*

$$C(N,\phi) = C(N, n\cdot\phi).$$

Proof . One has $\tau^{(2)}(N, n\cdot\phi)(t) = \tau^{(2)}(N,\phi)(t^n)$ from the definition of L^2-Alexander torsion. $\square$

5.3. Upper bound for the leading coefficient

The goal of the remainder of this chapter is to study the invariant $C(N, \phi)$ in terms of the relative torsion $\tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_-)$. Recall that $\tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_-) = \tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_+)$ by Lemma 4.4. The main result of this section is the following theorem.

Theorem 5.11. *Let N be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary. Furthermore, let $\phi \in H^1(N;\mathbb{Z})$ be non-zero and let Σ be a Thurston norm minimizing surface dual to ϕ. Then the following inequality holds:*

$$C(N, \phi) \leqslant \tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_+) = \tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_-).$$

As the reader will notice the proof of this theorem is very technical. Before we give a proof in the next section, we sketch a moral argument, why such an inequality should hold and why one should expect even equality (see Conjecture 5.16). We further simplify to the case that Σ is connected and pick a loop $\gamma \in \pi_1(N)$ with $\phi(\gamma) = 1$. As usual, we abbreviate $\pi_1(N)$ by π.

One can compare the two short exact sequences of chain complexes:

$$0 \to C_*^{cw}(\Sigma; \mathcal{N}(\pi)^{(\phi,t)}) \xrightarrow{\gamma t i_+ - i_-} C_*^{cw}(N \setminus\!\setminus \Sigma; \mathcal{N}(\pi)^{(\phi,t)}) \longrightarrow C_*^{cw}(N; \mathcal{N}(\pi)^{(\phi,t)}) \longrightarrow 0,$$

$$0 \longrightarrow C_*^{cw}(\Sigma; \mathcal{N}(\pi)) \xrightarrow{\quad i_- \quad} C_*^{cw}(N \setminus\!\setminus \Sigma; \mathcal{N}(\pi)) \longrightarrow C_*^{cw}(N \setminus\!\setminus \Sigma, \Sigma_-; \mathcal{N}(\pi)) \to 0.$$

Note that $\pi_1(\Sigma)$ and $\pi_1(N \setminus\!\setminus \Sigma)$ are in the kernel of $\phi\colon \pi_1(N) \to \mathbb{Z}$. This yields the following equalities of $\mathcal{N}(\pi)$-modules:

$$C_*^{cw}(\Sigma; \mathcal{N}(\pi)^{(\phi,t)}) = C_*^{cw}(\Sigma; \mathcal{N}(\pi)) \quad \text{and} \quad C_*^{cw}(N \setminus\!\setminus \Sigma; \mathcal{N}(\pi)^{(\phi,t)}) = C_*^{cw}(N \setminus\!\setminus \Sigma; \mathcal{N}(\pi)).$$

We abbreviate $T_\Sigma = \tau^{(2)}(C_*^{cw}(\Sigma; \mathcal{N}(\pi))$ and $T_{N \setminus\!\setminus \Sigma} = \tau^{(2)}(C_*^{cw}(N \setminus\!\setminus \Sigma; \mathcal{N}(\pi)))$. We apply Lemma 2.24 to both of the above short exact sequences of chain complexes and compare the results:

$$\tau^{(2)}(N, \phi)(t) = \frac{T_{N \setminus\!\setminus \Sigma}}{T_\Sigma} \cdot \det_{\mathcal{N}(\pi)}\big(\gamma t \cdot i_+ - i_-\colon\ H_1^{(2)}(\Sigma) \to H_1^{(2)}(N \setminus\!\setminus \Sigma)\big),$$
$$\tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_-) = \frac{T_{N \setminus\!\setminus \Sigma}}{T_\Sigma} \cdot \det_{\mathcal{N}(\pi)}\big(i_-\colon\ H_1^{(2)}(\Sigma) \to H_1^{(2)}(N \setminus\!\setminus \Sigma)\big).$$

We see they differ by the value of $\det_{\mathcal{N}(\pi)}(\gamma t^{\phi(\gamma)} \cdot i_+ - i_-)$ and $\det_{\mathcal{N}(\pi)}\big(i_-\big)$. Moreover, in view of Lemma 5.8 the leading coefficient is defined morally by

$$C(N, \phi) = \lim_{t \to 0} \tau^{(2)}(N, \phi)(t).$$

If we would know that $\lim_{t \to 0} \det_{\mathcal{N}(G)}(\gamma t^{\phi(\gamma)} \cdot i_+ - i_-) = \det_{\mathcal{N}(G)}\big(i_-\big)$, then this would imply that the leading coefficient is equal to the relative torsion.

We recap the problems with this approach. First, it is false in general that the Fuglede-Kadison determinant is continuous as we have seen in Example 2.17. Secondly, the equation $C(N, \phi) = \lim_{t \to 0} \tau^{(2)}(N, \phi)(t)$ only holds for a special representative of the L^2-Alexander torsion and therefore one has to be very careful about the choice of CW-structure.

The proof of the inequality

We now prove Theorem 5.11. We proceed in two steps. We first prove the statement for closed N and then by a doubling argument we extend it to the case of toroidal boundary.

Proposition 5.12. *Let N be a connected closed oriented irreducible 3-manifold. Furthermore, let $\phi \in H^1(N; \mathbb{Z})$ be non-zero and let Σ be a Thurston norm minimizing surface dual to ϕ. We have*

$$C(N, \phi) \leqslant \tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_+) = \tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_-).$$

One of the main ingredients in the proof is to find a suitable CW-structure. This is the content of the next lemma.

Lemma 5.13. *Let N be a connected closed oriented irreducible 3-manifold and S' an embedded surface such that $N \setminus\setminus S'$ is connected. Then there is a CW-structure for N with the following properties:*

1. *$M := N \setminus S' \times (-1, 1)$ and $S' \times [-1, 1]$ are subcomplexes,*
2. *the CW-structure on $S' \times [-1, 1]$ is a product structure,*
3. *M has precisely one 3-cell β,*
4. *there is exactly one 0-cell q in the interior $M \setminus S' \times \{\pm 1\}$,*
5. *S' has only one 0-cell p_i in each component S_i,*
6. *for each i there exist 1-cells $v_i^\pm$ going from q to $p_i^\pm$ lying completely in M.*

Sketch of the argument . We pick a triangulation for $M = N \setminus S' \times (-1, 1)$. Since all triangulations on surfaces are equivalent after isotopies and subdivisions we can find a triangulation for M such that the triangulations on the two copies $S' \times \{\pm 1\}$ in M agree. We use this triangulation to view M as a CW-complex and we also equip S' as a CW-complex coming from the triangulation. Next we modify the CW-structure to also obtain properties (3), (4), (5) and (6). We do so following an argument of McMullen, see [McM02, Proof of Theorem 5.1]:

(a) Since our CW-structure comes from a triangulation it is well known that one can start with a top dimensional simplex and let it swallow all other simplexes to achieve (3).

(b) We pick a maximal tree in the 1-skeleton on M with the following properties:
   (i) the tree connects all vertices in $M \setminus S'_\pm$,
   (ii) the tree lies in $M \setminus S'_\pm$.

   We collapse this tree to a single point q. Since any embedded tree in a 3-manifold has a neighbourhood that is a ball we see that the collapsed space is again homeomorphic to M. But now we have a CW-structure that also satisfies (4) and (6).

(c) Finally for each component S_i of S' we pick a maximal tree T_i in the 1-skeleton of S_i that connects all vertices. We collapse $T_i \times [-1, 1]$. Once again the quotient space is homeomorphic to M and this time we have a CW-structure that has all the desired properties. □

Proof of Proposition 5.12 . So let N be a connected closed oriented irreducible 3-manifold and $\phi \in H^1(N; \mathbb{Z})$ non-zero. Furthermore, let Σ be a Thurston norm minimizing surface dual to ϕ.

By Proposition 4.12 there are connected components $S_1, \ldots, S_n$ of Σ and positive numbers $w_1, \ldots, w_n$ with

1. the class $\sum_{i=1}^n w_i \cdot [S_i] = [\Sigma] \in H_2(N)$ is dual to ϕ,
2. we have $\sum_{i=1}^n -w_i \cdot \chi(S_i) = \|\phi\|$,
3. if we set $S' = \cup_{i=1}^n S_i$, then $M := N \setminus\setminus S'$ is connected,
4. and $\tau^{(2)}(M, S'_-) = \tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_-)$.

Recall that by Lemma 4.4 we have $\tau^{(2)}(M, S'_-) = \tau^{(2)}(M, S'_+)$. Therefore, it is sufficient to show the following claim.

Claim. The inequality $C(N, \phi) \leqslant \tau^{(2)}(M, S'_+) = \tau^{(2)}(M, S'_-) = \tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_-)$ holds.

We start by fixing a lot of notations. We denote by $p \colon \widetilde{N} \to N$ the universal covering. For any subset $X \subset N$ we write $\widetilde{X} := p^{-1}(X)$.

For S' in N we pick the CW structure from Lemma 5.13. Now we name the cells of $S' = S_1 \sqcup \ldots \sqcup S_n$:

$$\begin{aligned} \mathcal{P} &= \{p_i\}_{i \in \{1,\ldots,n\}} && \text{0-cells,} \\ \mathcal{E} &= \{e_i\}_{i \in \{1,\ldots,m\}} && \text{1-cells,} \\ \mathcal{S} &= \{s_i\}_{i \in \{1,\ldots,k\}} && \text{2-cells.} \end{aligned}$$

We write $I = [-1, 1]$ for the interval. We equip $S' \times I$ with the product CW-structure and denote the set of cells by

$$\begin{aligned} \mathcal{P}^\pm &= \left\{p_i^\pm\right\}_{i \in \{1,\ldots,n\}}, & \mathcal{P} \times I &= \{p_i \times I\}_{i \in \{1,\ldots,n\}}, \\ \mathcal{E}^\pm &= \left\{e_i^\pm\right\}_{i \in \{1,\ldots,k\}}, & \mathcal{E} \times I &= \{e_i \times I\}_{i \in \{1,\ldots,k\}}, \\ \mathcal{S}^\pm &= \left\{s_i^\pm\right\}_{i \in \{1,\ldots,m\}}, & \mathcal{S} \times I &= \{s_i \times I\}_{i \in \{1,\ldots,m\}}. \end{aligned}$$

The CW-structure on M has $2n+1$ 0-cells, namely q and the $p_i^{\pm}$. From Lemma 5.13 we have distinguished 1-cells $\mathcal{V}^{\pm} = \left\{v_i^{\pm}\right\}$. Let $\mathcal{E}'$ and $\mathcal{S}'$ be the set of the other 1- and 2-cells in the interior of M.

We obtain an element γ_i in $\pi_1(N, q)$ by concatenating the paths v_i^-, $p_i \times I$ and $\overline{(v_i^+)}$, where $\overline{(v_i^+)}$ means the path in opposite direction. We have $\phi(\gamma_i) = w_i$ for all $i = 1, \ldots, n$.

We fix once and for all a lift $\widetilde{q} \in \widetilde{N}$ of q. Using $\widetilde{q}$ we choose specific lifts of the cells of $S \times I$. We lift the cells of $\mathcal{P}^+, \mathcal{E}^+$ and $\mathcal{S}^+$ (resp. $\mathcal{P}^-, \mathcal{E}^-$ and $\mathcal{S}^-$) by using the path v_i^+ (resp. v_i^-). For the cells $\mathcal{P} \times I$, $\mathcal{E} \times I$ and $\mathcal{S} \times I$ we choose the path v_i^-. Note that this choice later plays a role in the calculation of the boundary maps.

All remaining cells are touching cells that we already lifted. So for these cells there is a unique choice of lifts, which is coherent with the other lifts. As one might already suspect we write $\widetilde{\mathcal{V}^{\pm}}, \widetilde{\mathcal{E}'}$ and $\widetilde{\mathcal{S}'}$ for these lifts.

As a short recap we have that $C_*(\widetilde{N})$ is of the form:

$$\begin{aligned}
C_3(\widetilde{N}) &= \mathbb{Z}[\pi_1(N, q)] \cdot \langle \widetilde{\beta}, \widetilde{\mathcal{S} \times I} \rangle, \\
C_2(\widetilde{N}) &= \mathbb{Z}[\pi_1(N, q)] \cdot \langle \widetilde{\mathcal{S}'}, \widetilde{\mathcal{S}^+}, \widetilde{\mathcal{S}^-}, \widetilde{\mathcal{E} \times I} \rangle, \\
C_1(\widetilde{N}) &= \mathbb{Z}[\pi_1(N, q)] \cdot \langle \widetilde{\mathcal{P} \times I}, \widetilde{\mathcal{V}^+}, \widetilde{\mathcal{V}^-}, \widetilde{\mathcal{E}'}, \widetilde{\mathcal{E}^+}, \widetilde{\mathcal{E}^-} \rangle, \\
C_0(\widetilde{N}) &= \mathbb{Z}[\pi_1(N, q)] \cdot \langle \widetilde{\mathcal{P}^+}, \widetilde{\mathcal{P}^-}, \widetilde{q} \rangle,
\end{aligned}$$

where $\mathbb{Z}[\pi_1(N, q)] \cdot \langle B \rangle$ means the free $\mathbb{Z}[\pi_1(N, q)]$-module over the set B.

Next we give the boundary matrices with respect to the above direct sum decomposition. Notation and justifications are given below.

$$\partial_3^N := \begin{array}{cc}
 & \begin{array}{cccc} \widetilde{\mathcal{S}'} & \widetilde{\mathcal{S}^+} & \widetilde{\mathcal{S}^-} & \widetilde{\mathcal{E} \times I} \end{array} \\
\begin{array}{c} \widetilde{\beta} \\ \widetilde{\mathcal{S} \times I} \end{array} & \begin{pmatrix} A & -\mathbf{1} & \mathbf{1} & 0 \\ 0 & d_{\mathcal{S}}(\boldsymbol{\gamma}) & -\mathrm{Id} & \partial_2^{S'} \times I, \end{pmatrix}
\end{array}$$

$$\partial_2^N := \begin{array}{cc}
 & \begin{array}{cccccc} \widetilde{\mathcal{P} \times I} & \widetilde{\mathcal{V}^+} & \widetilde{\mathcal{V}^-} & \widetilde{\mathcal{E}'} & \widetilde{\mathcal{E}^+} & \widetilde{\mathcal{E}^-} \end{array} \\
\begin{array}{c} \widetilde{\mathcal{S}'} \\ \widetilde{\mathcal{S}^+} \\ \widetilde{\mathcal{S}^-} \\ \widetilde{\mathcal{E} \times I} \end{array} & \begin{pmatrix} 0 & B^+ & B^- & C & D^+ & D^- \\ 0 & 0 & 0 & 0 & \partial_2^{S',+} & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_2^{S',-} \\ \partial_1^{S'} \times I & 0 & 0 & 0 & d_{\mathcal{E}}(\boldsymbol{\gamma}) & -\mathrm{Id} \end{pmatrix}
\end{array}$$

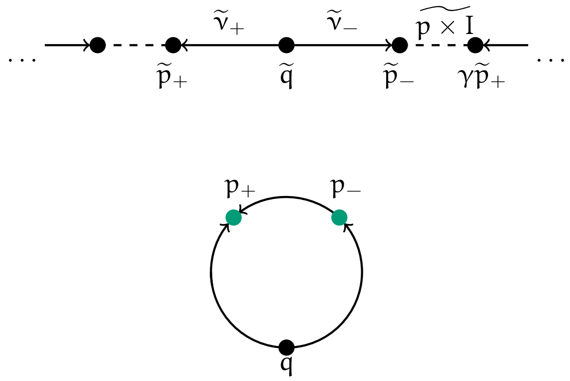

Figure 5.1.: This picture illustrates our convention of lifts of cells. Moreover, one can see that with these conventions one has $\partial(\widetilde{p \times I}) = \gamma \cdot \widetilde{p_+} - \widetilde{p_-}$.

$$\partial_1^N := \begin{array}{c} \\ \widetilde{\mathcal{P} \times I} \\ \widetilde{\mathcal{V}^+} \\ \widetilde{\mathcal{V}^-} \\ \widetilde{\mathcal{E}'} \\ \widetilde{\mathcal{E}^+} \\ \widetilde{\mathcal{E}^-} \end{array} \begin{array}{c} \begin{array}{ccc} \widetilde{\mathcal{P}^+} & \widetilde{\mathcal{P}^-} & \widetilde{q} \end{array} \\ \begin{pmatrix} d_{\mathcal{P}}(\boldsymbol{\gamma}) & -\mathrm{Id} & 0 \\ \mathrm{Id} & 0 & -\mathbf{1} \\ 0 & \mathrm{Id} & -\mathbf{1} \\ E^+ & E^- & F \\ \partial_1^{S',+} & 0 & 0 \\ 0 & \partial_1^{S',-} & 0 \end{pmatrix} \end{array}$$

Let us first settle some notation. We denote by $\mathbf{1}$ the matrix, where each entry is 1. We define $-\mathbf{1}$ in the same way. The symbols $A, B^{\pm}, C, D$ and $E^{\pm}$ are matrices of appropriate size. These are a priori matrices over $\mathbb{Z}[\pi_1(N, q)]$. But since S' is Thurston norm minimizing, the inclusion induced map $\pi_1(M, q) \to \pi_1(N, q)$ is a monomorphism (Theorem 1.33) and hence we can view them as matrices over $\mathbb{Z}[\pi_1(M, q)]$, because all corresponding cells are in the interior of M.

We write $\partial_*^{S \times I}$ for the boundary map of $C_*(\widetilde{S \times I})$ and $\partial_*^{S^{\pm}}$ for the restrictions to the sub-complexes $\widetilde{S^{\pm}}$.

Finally, we write $d_{\mathcal{S}}(\boldsymbol{\gamma})$ for the diagonal matrix indexed by $\mathcal{S}$. If $s_j \in \mathcal{S}$ is a cell of the connected component S_i, then the jth-entry of $d_{\mathcal{S}}(\boldsymbol{\gamma})$ is γ_i. The matrices $d_{\mathcal{E}}(\boldsymbol{\gamma})$ and $d_{\mathcal{P}}(\boldsymbol{\gamma})$ are defined analogously. Moreover, these entries need some explanation. If we consider a cell $\widetilde{s_i \times I} \in \widetilde{\mathcal{S} \times I}$, then recall that $\widetilde{s_i^-}$ and $\widetilde{s_i \times I}$ are lifts constructed using the path ν_i^- while $\widetilde{s_i^+}$ is a lift constructed using the path ν_i^+. But now the plus-boundary of the cell $\widetilde{s_i \times I}$ is the lift of s_i^+ coming from ν_i^- concatenated with $p_i \times I$. This lift differs from $\widetilde{s_i^+}$ by γ_i. We refer to Figure 5.1 for an illustration of this argument.

For the L^2-Alexander torsion we are interested in the matrices

$$\partial_*^{N,\phi,t} := \mathrm{Id}_{\mathcal{N}(\pi)} \otimes_{(\phi,t)} \partial_*.$$

Note that by our constructions we have $\phi(\gamma_i) = w_i$ for all $i \in \{1,\dots,n\}$ and $\pi_1(M,q) \subset \ker\phi$. Hence for the boundary matrices $\partial_j^{N,\phi,t}$ ($j = 1,2,3$) of the L^2-chain complex $C_*(N;\mathcal{N}(\pi)^{\phi,t})$ the only thing that changes is that γ_i is replaced $t^{w_i} \cdot \gamma_i$ for all $i \in \{1,\dots,n\}$. We denote the resulting matrices by $d_{\mathcal{S}}(t^{\boldsymbol{w}} \cdot \boldsymbol{\gamma})$, $d_{\mathcal{E}}(t^{\boldsymbol{w}} \cdot \boldsymbol{\gamma})$ and $d_{\mathcal{P}}(t^{\boldsymbol{w}} \cdot \boldsymbol{\gamma})$.

The above CW-structure also gives a CW-structure for the pair (M, S'_-) and we obtain the cellular chain complex $C_*(\widetilde{M}, \widetilde{S'_-})$:

$$\begin{aligned}
C_3(\widetilde{M}, \widetilde{S'_-}) &= \Lambda'\langle\widetilde{\beta}\rangle,\\
C_2(\widetilde{M}, \widetilde{S'_-}) &= \Lambda'\langle\widetilde{\mathcal{F}'}, \widetilde{\mathcal{F}^+}\rangle,\\
C_1\left(\widetilde{M}, \widetilde{S'_-}\right) &= \Lambda'\langle\widetilde{\mathcal{V}^+}, \widetilde{\mathcal{V}^-}, \widetilde{\mathcal{E}'}, \widetilde{\mathcal{E}^+}\rangle,\\
C_0(\widetilde{M}, \widetilde{\Sigma_-}) &= \Lambda'\langle\widetilde{\mathcal{P}^+, q}\rangle,
\end{aligned}$$

where the boundary matrices are given by

$$\partial_3^{\widetilde{M}} := \begin{array}{cc} & \begin{array}{cc} \widetilde{\mathcal{S}'} & \widetilde{\mathcal{S}^+} \end{array} \\ \widetilde{\beta} & \begin{pmatrix} A & -1 \end{pmatrix} \end{array},$$

$$\partial_2^{\widetilde{M}} := \begin{array}{cc} & \begin{array}{cccc} \widetilde{\mathcal{V}^+} & \widetilde{\mathcal{V}^-} & \widetilde{\mathcal{E}'} & \widetilde{\mathcal{E}^+} \end{array} \\ \begin{array}{c} \widetilde{\mathcal{S}'} \\ \widetilde{\mathcal{S}^+} \end{array} & \begin{pmatrix} B^+ & B^- & C & D^+ \\ 0 & 0 & 0 & \partial^2_{S',+} \end{pmatrix} \end{array},$$

$$\partial_1^{\widetilde{M}} := \begin{array}{cc} & \begin{array}{cc} \widetilde{\mathcal{P}^+} & \widetilde{q} \end{array} \\ \begin{array}{c} \widetilde{\mathcal{V}^+} \\ \widetilde{\mathcal{V}^-} \\ \widetilde{\mathcal{E}'} \\ \widetilde{\mathcal{E}^+} \end{array} & \begin{pmatrix} \mathrm{Id} & -1 \\ 0 & -1 \\ E^+ & F \\ \partial^1_{S',+} & 0 \end{pmatrix} \end{array}.$$

Using this CW-structure we can compute $\tau^{(2)}(M, S'_-)$. But first we introduce a slight extension to the notation of Lemma 2.30. Given a matrix A and a set of columns J and rows L, we define

$$\begin{aligned}
A[J,] &= \text{submatrix given by the columns corresponding to } J,\\
A[,L] &= \text{submatrix given by the rows corresponding to } L,\\
A[\widehat{J},] &= \text{submatrix given by deleting the columns corresponding to } J,\\
A[,\widehat{L}] &= \text{submatrix given by deleting the row corresponding to } L.
\end{aligned}$$

If we now apply Lemma 2.30 to the chain complex

$$C_*(M, S'_-; \mathcal{N}(\pi)) \quad \text{with } J = \left(\widetilde{s_k^+}\right) \text{ and } L = \left(\widetilde{V^+}, \widetilde{v_1^-}\right),$$

then we get the following formula for the L^2-torsion of $C_*(M, S'_-; \mathcal{N}(\pi))$:

$$\tau^{(2)}(M, S'_-) = \frac{\det^r_{\mathcal{N}(\pi)}\left(\partial_2^{\widetilde{M}}[\widehat{L}, \widehat{J}]\right)}{\det^r_{\mathcal{N}(\pi)}\left(\partial_3^{\widetilde{M}}[, J]\right) \cdot \det^r_{\mathcal{N}(\pi)}\left(\partial_1^{\widetilde{M}}[L,]\right)}.$$

The concrete matrices are given by

$$\partial_3^M = \begin{array}{c} \\ \widetilde{\beta} \end{array} \begin{array}{c} \begin{array}{cccc} \widetilde{\mathcal{S}'} & \widetilde{s_1^+} & \dots & \widetilde{s_k^+} \end{array} \\ \begin{pmatrix} A & -1 & \dots & \boxed{-1} \end{pmatrix} \end{array} \; \partial_3^{\widetilde{M}}[, J] \, ,$$

$$\color{teal}{\partial_2^{\widetilde{M}}[\widehat{J}, \widehat{L}]} = \begin{array}{c} \\ \widetilde{\mathcal{S}'} \\ \widetilde{s_1^+} \\ \vdots \\ \widetilde{s_k^+} \end{array} \begin{array}{c} \begin{array}{ccccc} \widetilde{\mathcal{V}^+} & \widetilde{v_1^-} & \dots \widetilde{v_n^-} & \widetilde{\mathcal{E}'} & \widetilde{\mathcal{E}^+} \end{array} \\ \begin{pmatrix} B^+ & B^-[,1] & B^-[,\widehat{1}] & C & D^+ \\ 0 & 0 & 0 & 0 & \partial_2^{S',+}[,\widehat{k}] \\ \vdots & \vdots & & & \\ 0 & 0 & \dots & 0 & \partial_2^{S',+}[,\widehat{k}] \end{pmatrix} \end{array},$$

$$\partial_1^M = \begin{array}{c} \\ \widetilde{\mathcal{V}^+} \\ \widetilde{v_1^-} \\ \vdots \\ \widetilde{v_k^-} \\ \widetilde{\mathcal{E}'} \\ \widetilde{\mathcal{E}^+} \end{array} \begin{array}{c} \begin{array}{cc} \widetilde{\mathcal{P}^+} & \widetilde{q} \end{array} \; \partial_1^{\widetilde{M}}[L,] \\ \begin{pmatrix} \boxed{\begin{matrix} \mathrm{Id} & -1 \\ 0 & -1 \end{matrix}} \\ \begin{matrix} \vdots & \vdots \\ 0 & -1 \\ E^+ & F \\ \partial^1_{\Sigma,+} & 0 \end{matrix} \end{pmatrix} \end{array}.$$

and hence we obtain the formula

$$\tau^{(2)}(M, S'_-) = \frac{\det^r_{G'}\begin{pmatrix} B^-[,\widehat{1}] & C & D^+ \\ 0 & 0 & \partial_2^{S',+}[,\widehat{k}] \end{pmatrix}}{\det^r_{G'}(-1) \cdot \det^r_{G'}\begin{pmatrix} \mathrm{Id} & -1 \\ 0 & -1 \end{pmatrix}} = \det^r_{G'}\begin{pmatrix} B^-[,\widehat{1}] & C & D^+ \\ 0 & 0 & \partial_2^{S',+}[,\widehat{k}] \end{pmatrix}. \tag{5.1}$$

Using the complete same lines of arguments one can also compute

$$\tau^{(2)}(M, S'_+) = \det^r_{G'}\begin{pmatrix} B^+[,\widehat{n}] & C & D^- \\ 0 & 0 & \partial_2^{S',-}[,\widehat{1}] \end{pmatrix}. \tag{5.2}$$

We will use the two calculations to compute the leading coefficient. Therefore, let

$$f(t) := \tau^{(2)}\left(C_*^{(2)}(N,\phi,t)\right)$$

be the particular representative of the L^2-Alexander torsion $\tau^{(2)}(N,\phi,t)$ computed from the previous choices of cells and lifts. To describe $f(t)$ we again use Lemma 2.30 applied to the chain complex

$$C_*(N;\mathcal{N}(\pi)^{\phi,t}) \text{ with } J = \left\{\widetilde{s_k^+},\widetilde{\mathcal{S}^-}\right\} \text{ and } L = \left(\widetilde{\mathcal{P}\times I},\widetilde{\mathcal{V}^+},\widetilde{\nu_1^-}\right).$$

Then we have

$$f(t) = \frac{\det^{\mathbf{r}}_{\mathcal{N}(\pi)}\left(\partial_2^{N,\phi,t}[\widehat{L},\widehat{J}]\right)}{\det^{\mathbf{r}}_{\mathcal{N}(\pi)}\left(\partial_3^{N,\phi,t}[L,]\right)\cdot\det^{\mathbf{r}}_{\mathcal{N}(\pi)}\left(\partial_1^{N,\phi,t}[,J]\right)}.$$

Below we illustrate these three matrices and compute the torsion as far as possible. We start with:

$$\partial_3^N := \begin{array}{c} \\ \widetilde{\beta} \\ \widetilde{\mathcal{F}\times I} \end{array} \begin{array}{ccccc} \widetilde{\mathcal{S}} & \dots & \widetilde{f_l^+} & \widetilde{\mathcal{F}^-} & \widetilde{\mathcal{E}\times I} \\ \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccccc} \widetilde{\mathcal{S}} & \dots & \widetilde{f_l^+} & \widetilde{\mathcal{F}^-} & \widetilde{\mathcal{E}\times I} \end{array} \\ \begin{array}{c} \widetilde{\beta} \\ \\ \widetilde{\mathcal{F}\times I} \end{array} & \left(\begin{array}{ccccc} A & \dots & -1 & \mathbf{1} & 0 \\ & & 0 & & \\ 0 & \dots & \vdots & -\mathrm{Id} & \partial_\Sigma^2\times I \\ & & t^{w_n}\gamma_n & & \end{array}\right) \end{array}.$$

$$\partial_3^{N,\phi,t}[L,]$$

Moreover, we can use elementary row and column operations and Lemma 2.19 to compute

$$\det^{\mathbf{r}}_{\mathcal{N}(\pi)}\left(\partial_3^{N,\phi,t}[L,]\right) = \det^{\mathbf{r}}_{\mathcal{N}(\pi)}\begin{pmatrix} -1 & 1 & \dots & 1 \\ 0 & & & \\ \vdots & & & \\ 0 & & -\mathrm{Id} & \\ t^{w_n}\cdot\gamma_n & & & \end{pmatrix} = \max\{1,t^{w_n}\}.$$

For the next matrix $\partial_2^{N,\phi,t}[\widehat{L},\widehat{J}]$ we have

$$\begin{array}{cc} & \begin{array}{cccccccc} \widetilde{\mathcal{P}\times I} & \widetilde{\mathcal{V}^+} & \widetilde{\nu_1^-} & \dots\widetilde{\nu_n^-} & \widetilde{\mathcal{M}} & \widetilde{\mathcal{E}^+} & \widetilde{\mathcal{E}^-} \end{array} \\ \begin{array}{c} \widetilde{\mathcal{S}} \\ \vdots \\ \widetilde{f_l^+} \\ \widetilde{\mathcal{F}^-} \\ \widetilde{\mathcal{E}\times I} \end{array} & \left(\begin{array}{ccccccc} 0 & B^+ & B^-[,1] & B^-[,\widehat{1}] & C & D^+ & D^- \\ \vdots & \vdots & \vdots & 0 & 0 & \partial_2^{S,+}[\widehat{1},] & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_2^{S,+}[1,] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial^2_{\Sigma,-} \\ \partial_\Sigma^1\times I & 0 & 0 & 0 & 0 & d_{\mathcal{E}}(t^w\gamma) & -\mathrm{Id} \end{array}\right) \end{array}$$

and hence

$$\det\nolimits^{r}_{\mathcal{N}(\pi)}\left(\partial_2^{N,\phi,t}[\widehat{L},\widehat{J}]\right) = \det\nolimits^{r}_{\mathcal{N}(\pi)}\begin{pmatrix} B^-[,\widehat{1}] & C & D^+ & D^- \\ 0 & 0 & \partial_S^+[,\widehat{l}] & 0 \\ 0 & 0 & d_{\mathcal{E}}(t^w\gamma) & -\operatorname{Id} \end{pmatrix}.$$

The last matrix is given by

$$\partial_1^N := \begin{array}{c} \\ \widetilde{\mathcal{P}\times I} \\ \widetilde{\mathcal{V}^+} \\ \widetilde{v_1^-} \\ \vdots \\ \widetilde{\mathcal{M}} \\ \widetilde{\mathcal{E}^+} \\ \widetilde{\mathcal{E}^-} \end{array} \begin{array}{c} \begin{array}{ccc} \widetilde{\mathcal{P}^+} & \widetilde{\mathcal{P}^-} & \widetilde{q} \quad \partial_1^{N,\phi,t}[,J] \end{array} \\ \left(\begin{array}{ccc} d_{\mathcal{P}}(t^w\gamma) & -\operatorname{Id} & 0 \\ \operatorname{Id} & 0 & -1 \\ 0 & 1\,0\ldots0 & -1 \\ \vdots & \vdots & \vdots \\ E^+ & E^- & F \\ \partial^1_{\Sigma,+} & 0 & 0 \\ 0 & \partial^1_{\Sigma,-} & 0 \end{array}\right) \end{array}$$

And we can again use elementary row and column operations and Lemma 2.19 to conclude

$$\begin{aligned} \det\nolimits^{r}_{\mathcal{N}(\pi)}\left(\partial_1^{N}[,J]\right) &= \det\nolimits^{r}_{\mathcal{N}(\pi)}\begin{pmatrix} d_{\mathcal{P}}(t^w\gamma) & -\operatorname{Id} & 0 \\ \operatorname{Id} & 0 & -1 \\ 0 & 1\,0\ldots0 & -1 \end{pmatrix} \\ &= \max\{1,t\}^{w_1}. \end{aligned}$$

Putting everything together we have for our representative of the L^2-Alexander torsion

$$f(t) = \det\nolimits^{r}_{\mathcal{N}(\pi)}\begin{pmatrix} B[,\widehat{1}]^- & C & D^+ & D^- \\ 0 & 0 & \partial_S^+[,\widehat{l}] & 0 \\ 0 & 0 & d_{\mathcal{E}}(t^w\cdot\gamma) & -\operatorname{Id} \end{pmatrix}\cdot\max\{1,t\}^{-w_1-w_n}.$$

Let us assume for a second that $f(t)$ is the representative of Lemma 5.8, so that the leading coefficient is given by

$$C(N,\phi) = \lim_{t\to 0^+} f(t) = \lim_{t\to 0^+}\det\nolimits^{r}_{\mathcal{N}(\pi)}\begin{pmatrix} B[,\widehat{1}]^- & C & D^+ & D^- \\ 0 & 0 & \partial_S^+[,\widehat{l}] & 0 \\ 0 & 0 & d_{\mathcal{E}}(t^w\cdot\gamma) & -\operatorname{Id} \end{pmatrix}.$$

In this case Lemma 2.20 would yield

$$
\begin{aligned}
C(N,\phi) &= \lim_{t\to 0^+} \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)} \begin{pmatrix} B[,\widehat{1}]^- & C & D^+ & D^- \\ 0 & 0 & \partial_S^+[,\widehat{l}] & 0 \\ 0 & 0 & d_{\mathcal{E}}(t^{w}\cdot\boldsymbol{\gamma}) & -\operatorname{Id} \end{pmatrix} \\
&\leqslant \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)} \begin{pmatrix} B[,\widehat{1}]^- & C & D^+ & D^- \\ 0 & 0 & \partial_S^+[,\widehat{l}] & 0 \\ 0 & 0 & 0 & -\operatorname{Id} \end{pmatrix} \\
&= \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}(-\operatorname{Id}) \cdot \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)} \begin{pmatrix} B[,\widehat{1}]^- & C & D^+ \\ 0 & 0 & \partial_S^+[,\widehat{l}] \end{pmatrix} = \tau^{(2)}(M, S'_+),
\end{aligned}
$$

which would finish the proof. Therefore, we are left to show that $f(t)$ is indeed the representative of Lemma 5.8 i. e. we have to show that $\lim_{t\to\infty} f(t)\cdot t^{-\|\phi\|}$ exists.

In order to archive this goal we are now going to calculate $f(t)$ differently. Let $\mathcal{P}_i, \mathcal{E}_i$ and $\mathcal{F}_i$ be the sets of 0,1 and 2-cells of S_i. We write

$$
|\mathcal{P}|_w := \sum_{i=1}^{l} w_i \cdot |\mathcal{P}_i| = \sum_{i=1}^{l} w_i, \qquad |\mathcal{E}|_w := \sum_{i=1}^{l} w_i \cdot |\mathcal{E}_i|, \qquad |\mathcal{S}|_w := \sum_{i=1}^{l} w_i \cdot |\mathcal{S}_i|.
$$

With this convenient notation one has $\|\phi\| = -|\mathcal{S}|_w + |\mathcal{E}|_w - |\mathcal{P}|_w$.

We again use Lemma 2.30 but now we apply it to $C_*(N; \mathcal{N}(\pi)^{\phi,t})$ with

$$
J' = \big(\widetilde{\mathcal{S}^+}, \widetilde{s_1^-}\big) \text{ and } L' = \big(\widetilde{\mathcal{P}\times I}, \widetilde{v_l^+}, \widetilde{\mathcal{V}^-}\big).
$$

We basically change the role of $+$ and $-$. One can see as before:

$$
\begin{aligned}
\det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\Big(\partial_3^{N,\phi,t}[J',]\Big) &= \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)} \begin{pmatrix} -1\ldots-1 & 1 \\ & -1 \\ & 0 \\ d_{\mathcal{S}}(t^{w}\cdot\boldsymbol{\gamma}) & \vdots \\ & 0 \end{pmatrix} \\
&= \frac{\max\{1,t\}^{w_1}}{t^{w_1}} \cdot \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\big(d_{\mathcal{S}}(t^{w}\boldsymbol{\gamma})\big) = \frac{\max\{1,t\}^{w_1}\cdot t^{|\mathcal{S}_w|}}{t^{w_1}},
\end{aligned}
$$

$$
\det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\Big(\partial_2^{N,\phi,t}[\widehat{L'},\widehat{J'}]\Big) = \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)} \begin{pmatrix} B^+[,\widehat{n}] & C & D^+ & D^- \\ 0 & 0 & 0 & \partial_2^{S',-}[\widehat{1},] \\ 0 & 0 & d_{\mathcal{E}}(t^{w}\cdot\boldsymbol{\gamma}) & -\operatorname{Id} \end{pmatrix},
$$

$$
\begin{aligned}
\det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\Big(\partial_1^{N,\phi,t}[,J']\Big) &= \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)} \begin{pmatrix} d_{\mathcal{P}}(t^{w}\cdot\boldsymbol{\gamma}) & -\operatorname{Id} & 0 \\ 0\ldots0\;1 & 0 & -1 \\ 0 & \operatorname{Id} & -1 \end{pmatrix} \\
&= \frac{\max\{1,t\}^{w_n}}{t^{w_n}} \cdot \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)} d_{\mathcal{P}}(t^{w}\boldsymbol{\gamma}) = \frac{\max\{1,t\}^{w_n}\cdot t^{|\mathcal{P}|_w}}{t^{w_n}}.
\end{aligned}
$$

The determinant of the second boundary matrix can be rewritten to

$$\begin{aligned}\det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\partial_2^{N,\phi,t}[\widehat{L'},\widehat{J'}] &= \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\begin{pmatrix} B^+[,\widehat{n}] & C & D^+ & D^- \\ 0 & 0 & 0 & \partial_2^{S',-}[\widehat{1},] \\ 0 & 0 & d_{\mathcal{E}}(t^{w}\gamma) & -\mathrm{Id}\end{pmatrix}\\ &= t^{|\mathcal{E}|_w}\cdot\det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\begin{pmatrix} B^+[,\widehat{n}] & C & D^+ & D^- \\ 0 & 0 & 0 & \partial_2^{S',-}[\widehat{1},] \\ 0 & 0 & -\mathrm{Id} & d_{\mathcal{E}}(t^{-w}\gamma)\end{pmatrix}.\end{aligned}$$

Using the equation $\|\phi\| = |\mathcal{E}|_w - |\mathcal{S}|_w - |\mathcal{P}|_w$ we obtain:

$$f(t) = \frac{t^{\|\phi\|}\cdot t^{w_1+w_n}}{\max\{1,t\}^{w_1+w_n}}\cdot\det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\begin{pmatrix} B^+[,\widehat{n}] & C & D^+ & D^- \\ 0 & 0 & 0 & \partial_2^{S',-}[\widehat{1},] \\ 0 & 0 & -\mathrm{Id} & d_{\mathcal{E}}(t^{-w}\cdot\gamma)\end{pmatrix}.$$

Using Lemma 2.20 and Equation 5.2 we conclude

$$\begin{aligned}\lim_{t\to\infty}\frac{f(t)}{t^{\|\phi\|}} &= \lim_{t\to\infty}\det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\begin{pmatrix} B^+[,\widehat{n}] & C & D^+ & D^- \\ 0 & 0 & 0 & \partial_2^{S',-}[\widehat{1},] \\ 0 & 0 & -\mathrm{Id} & d_{\mathcal{E}}(t^{-w}\cdot\gamma)\end{pmatrix}\\ &\leqslant \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\begin{pmatrix} B^+[,\widehat{n}] & C & D^+ & D^- \\ 0 & 0 & 0 & \partial_2^{S',-}[\widehat{1},] \\ 0 & 0 & -\mathrm{Id} & 0\end{pmatrix}\\ &= \det{}^{\mathbf{r}}_{\mathcal{N}(\pi)}\begin{pmatrix} B^+[,\widehat{n}] & C & D^- \\ 0 & 0 & \partial_2^{S',-}[\widehat{1},]\end{pmatrix} = \tau^{(2)}(M,S'_-).\end{aligned}$$

Therefore, the limit $\lim\limits_{t\to\infty} f(t)\cdot t^{-\|\phi\|}$ exists and we see that $f(t)$ is indeed the representative of Lemma 5.8. This finally finishes the proof. $\square$

The extension of the last proposition to the case of non-empty toroidal boundary is a straight forward doubling argument. For the convenience of the reader we outline the steps.

Lemma 5.14. *Let* N *be a connected compact oriented irreducible* 3*-manifold with non-empty toroidal boundary. Furthermore, let* $\phi \in H^1(N;\mathbb{Z})$ *be non-zero and* Σ *be a Thurston norm minimizing surface dual to* ϕ*. We have*

$$C(N,\phi) \leqslant \tau^{(2)}(N\setminus\!\setminus\Sigma,\Sigma_-) = \tau^{(2)}(N\setminus\!\setminus\Sigma,\Sigma_-).$$

Proof . We take two disjoint copies N_1 and N_2 of N and consider the double $D(N)$ of N and the double $D(\Sigma)$ of Σ in $D(N)$ as in Lemma 1.11. By the same lemma we know that $D(\Sigma)$ is Thurston norm minimizing. We consider the obvious retraction $r\colon D(N)\to N$ from $D(N)$ onto N. So If $i_k\colon N\to D(N)$ $(k=1,2)$ are the obvious inclusions, then $r \circ i_k = \mathrm{Id}_N$. It is a small exercise in algebraic topology to show $r^*(\phi)$ is Poincaré dual to $D(\Sigma)$.

Since $D(N)$ is closed, we can apply the previous proposition and obtain

$$C(D(N), r^*(\phi)) \leqslant \tau^{(2)}(D(N) \setminus\!\setminus D(\Sigma), D(\Sigma_+)).$$

It follows from Proposition 4.13 that

$$\tau^{(2)}(D(N) \setminus\!\setminus D(\Sigma), D(\Sigma_-)) = \tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_-)^2$$

It follow from Lemma 5.9 that

$$\begin{aligned} C(D(N), r^*(\phi)) &= C(N_1, (r \circ i_1)^*(\phi)) \cdot C(N_2, (r \circ i_2)^*\phi) \\ &= C(N, \phi)^2. \end{aligned}$$

In conclusion, we have $C(N, \phi)^2 \leqslant \tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_-)^2$ which after taking the square root finishes the proof. □

In the proof one saw that we only got an inequality, because the regular Fuglede-Kadison determinant is only upper semi continuous. We make the following conjecture, which is a weaker version of a question asked by Lück [Lü17, Question 9.11].

Conjecture 5.15. Let G be a group of class $\mathcal{G}$ and $A \in \mathcal{M}at_{n+k}(\mathcal{N}(G))$. Given positive numbers $w_1, \ldots, w_n$, we assume that for all $t \in \mathbb{R}_{>0}$ the matrix

$$A - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & t^{w_1} & \ldots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & t^{w_n} \end{pmatrix}$$

is a weak isomorphism. Suppose A is a weak isomorphism, then

$$\lim_{t \to 0} \det{}^{r}_{\mathcal{N}(G)} \left(A - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & t^{w_1} & \ldots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & t^{w_n} \end{pmatrix} \right) = \det{}^{r}_{\mathcal{N}(G)} A.$$

Note that this conjecture together with our proof of Theorem 5.11 suggest the following conjecture.

Conjecture 5.16. Let N be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary. Furthermore, let $\phi \in H^1(N; \mathbb{Z})$ be non-zero and Σ be a Thurston norm minimizing surface dual to ϕ. One has

$$C(N, \phi) = \tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_+) = \tau^{(2)}(N \setminus\!\setminus \Sigma, \Sigma_-).$$

CHAPTER 6

Some calculations

We end this thesis with some calculations of relative torsion and questions about the relation to geometric quantities. The calculations can already be found in a preprint [BFH18].

6.1. Ahyperbolic surfaces

Definition 6.1. We call a taut sutured manifold (M, R_+, R_-, γ) ahyperbolic if there is a disjoint union $\mathcal{C}$ of properly embedded incompressible tori $T_1, \ldots, T_n$ and annuli $A_1, \ldots, A_k$ in M with the following properties:

1. each annulus component A in $\mathcal{C}$ touches R_+ and R_- and is not boundary parallel,
2. each component M' of $M \setminus\setminus \mathcal{C}$ is, as a pair of spaces $(M', M' \cap R_-)$, homeomorphic to one of the following three simple types:
   a) (N, F) where N is a Seifert fibred space and F is a union of boundary tori and of π_1-injective annuli lying in the boundary,
   b) (V, C) where V is a solid torus and C is a collection of essential annuli in the boundary of V (here essential means π_1-injective in V),
   c) $(S \times I, S \times \{-1\})$, where S is a surface with boundary.

Definition 6.2. A taut surface Σ in a compact oriented 3-manifold N with empty or toroidal boundary is called ahyperbolic if $N \setminus\setminus \Sigma$, viewed as a sutured manifold, is ahyperbolic.

Proposition 6.3. *If* (M, R_+, R_-, γ) *is ahyperbolic, then*

$$\tau^{(2)}(M, R_-) = 1.$$

In particular, if Σ *in* N *is a ahyperbolic surface, then*

$$\tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_-) = 1.$$

Proof . We start the proof with the case that $\mathcal{C}$ is empty. So we look at the three types a), b) and c).

a) If N is a Seifert fibred space and F is a union of boundary tori and of π_1-injective annuli lying in the boundary, then we have $\tau^{(2)}(F \subset N) = 1$ by Lemma 2.32 and $\tau^{(2)}(N) = 1$ by Theorem 2.33. Thus by Lemma 2.28 one has $\tau^{(2)}(N, F) = 1$.

b) Let V be a solid torus and C be a collection of incompressible annuli in the boundary of V. We have $\tau^{(2)}(V) = 1$ and $\tau^{(2)}(C \subset V) = 1$ by Lemma 2.32. Then again by Lemma 2.28 we obtain $\tau^{(2)}(V, C) = 1$.

c) This follows from the homotopy invariance of the L^2-torsion (see Theorem 2.25).

The general case of this proposition i. e. the collection $\mathcal{C}$ is non-empty follows directly from the three calculation above and Lemma 4.2. □

In the remainder of this section we will give examples of ahyperbolic surfaces.

Proposition 6.4. *Let N be a graph manifold. Every taut surface Σ in N is ahyperbolic and hence*

$$\tau^{(2)}(N \setminus\setminus \Sigma, \Sigma_-) \ = \ 1.$$

Proof . First we assume that N is a Seifert fibred space. It is a classical result [Ja80, Theorem VI.34] that every Thurston norm minimizing surface is one of the following:

1. it is either a disjoint union of fibres of a fibration over S^1 or
2. it is a disjoint union of tori and annuli, each of which is saturated in the Seifert fibration.

In the first case $N \setminus\setminus \Sigma$ consists of products and hence are of type (c). In the other case $N \setminus\setminus \Sigma$ is of type (a). This proves the proposition for Seifert fibred spaces. Now let N be a graph manifold. By the definition of a graph manifold we find disjointly embedded incompressible tori $T_1, \ldots, T_n$ such that

$$N \setminus\setminus (T_1 \cup \ldots \cup T_n) = M_1 \sqcup \ldots \sqcup M_k$$

is the disjoint union of Seifert fibred spaces. We chose a minimal collection $\mathcal{T}$ of such tori. By Proposition 1.9 we can assume that Σ is in general position with $\mathcal{T}$ such that $\Sigma_i := \Sigma \cap M_i$ is Thurston norm minimizing for each $i \in \{1, \ldots, k\}$.

We define $\mathcal{C} := \mathcal{T} \cap N \setminus\setminus \Sigma$. This is a union of tori and annuli. Note that the connected components of $(N \setminus\setminus \Sigma) \setminus\setminus \mathcal{C}$ are given by $M_1 \setminus\setminus \Sigma_1, \ldots M_k \setminus\setminus \Sigma_k$. Since each M_i is Seifert fibred the statement follows from the first part of the proof once we showed that each Σ_i is a decomposition surface. Then each annulus component $A \subset \mathcal{C}$ touches Σ_+ and Σ_-.

Claim. Each Σ_i is a decomposition surface in M_i.

We assume the contrary. Then there is an embedded oriented annulus A in a connected component X of ∂M_i bounding parts of $\partial \Sigma_i$. By the classification of Thurston norm minimizing surfaces, this can only happen if Σ_i is a disjoint union of tori and annuli, each of which is saturated in the Seifert fibration. We see that $\partial \Sigma_i \cap X$ are fibres of the Seifert fibration. Note that X correspond to a connected component

T_k of $\mathcal{T}$. Without loss of generality we assume that $T_{k+} = X$. Let M_j be the Seifert fibred space containing T_{k-}. Then $\partial\Sigma_j \cap T_{k-}$ also bound an oriented annulus. Hence $\partial\Sigma_j \cap T_{k-}$ are fibres of the Seifert fibration of M_j. We conclude that $N \setminus\setminus (\mathcal{T} \setminus T_k)$ is a union of Seifert fibred spaces which contradict the minimality assumption on $\mathcal{T}$. This finishes the proof of the claim and hence the proof of this proposition. □

Agol and Dunfield [AD15] introduced the notation of a libroid knot. This is related to our concept in the following way. If a knot K is libroid, then by definition, there exists an $n \in \mathbb{N}$ such that $n \cdot \phi_K \in H^1(E_K;\mathbb{Z})$ is represented by a surface Σ which is ahyperbolic.

Agol and Dunfield proved that the class of libroid knots contains all 2-bridge knots [AD15, Section 6]. We can now combine this fact with Theorem 5.11, Theorem 5.7 (4a) and Proposition 6.3 to obtain:

Corollary 6.5. *There exist infinitely many non-fibred hyperbolic knots K in S^3 such that*

$$C(S^3 \setminus \nu(K), \phi_K) \;=\; 1,$$

where ϕ_K is a generator of $H^1(S^3 \setminus \nu(K);\mathbb{Z})$.

Relation to geometric quantities

We end this thesis with a conjecture which relates the relative L^2-torsion to a geometric quantity.

Conjecture 6.6. Let N be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary and Σ a taut surface in N. If $DM(\gamma)$ denotes the double of the sutured manifold $N \setminus\setminus\Sigma$, then

$$\tau^{(2)}(N \setminus\setminus\Sigma, \Sigma_-)^2 = \tau^{(2)}(N \setminus\setminus\Sigma, \Sigma_-) \cdot \tau^{(2)}(N \setminus\setminus\Sigma, \Sigma_+) = \tau^{(2)}(DM(\gamma)).$$

Recall that by Theorem 2.33 the value $\tau^{(2)}(DM(\gamma))$ is related to the volume of the hyperbolic pieces in the JSJ-decomposition. By Proposition 6.3 the conjecture is true for ahyperbolic surfaces. So the interesting case of this conjecture is the following.

Conjecture 6.7. If Σ is a taut and totally geodesic surface in a complete hyperbolic 3-manifold N of finite volume, then

$$\tau^{(2)}(N \setminus\setminus\Sigma, \Sigma_-) = e^{\mathrm{Vol}_{\mathbb{H}}(N)/6\pi}.$$

One way to tackle this conjecture is to use Lemma 2.24. Then one sees the following question arising.

Question 6.8. Let (M, R_+, R_-, γ) be a connected taut sutured manifold satisfying Assumption 4.0. Let M be endowed with a finite CW-structure such that R_+, R_- and γ are subcomplexes. Denote by $i_\pm\colon H_1(R_\pm;\mathcal{N}(\pi_1(M))) \to H_1(M;\mathcal{N}(\pi_1(M))$ the inclusion induced maps. What can we say about $\det_{\mathcal{N}(\pi_1(M))}(i_\pm)$?

Bibliography

[Ag08] I. Agol, *Criteria for virtual fibering*, J. Topol. 1, no. 2, 269–284 (2008)

[AD15] I. Agol and N. M. Dunfield, *Certifying the Thurston norm via* $\mathrm{SL}(2,\mathbb{C})$*-twisted homology*, to appear in the Thurston memorial conference proceedings, Princeton University Press, 23 pages (2015)

[Al28] J. W. Alexander, *Topological invariants of knots and links*, Trans. of the American Math. Soc., 30(2), 275–306 (1928)

[AFW15] M. Aschenbrenner, S. Friedl and H. Wilton, *3–manifold groups*, Lecture Notes of the European Math. Soc. (2015)

[At76] M. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Astérisque, 32(3), 43–72 (1976)

[BFH18] F. Ben-Aribi, S. Friedl and G. Herrmann, *The leading coefficient of the* L^2*-Alexander torsion*, preprint (2018)

[Br93] G.E. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics, Springer New York (1993)

[Ca07] D. Calegari, *Foliations and the Geometry of 3-Manifolds*, Oxford Mathematical Monographs, Clarendon Press (2007)

[De10] M. Dehn, *Über die Topologie des dreidimensionalen Raumes*, Mathematische Annalen, Volume 69(1), 137–168 (1910)

[Do79] J. Dodziuk, L^2*-harmonic forms on rotationally symmetric riemannian manifolds*, Proc. of the American Math. Soc., 395–400 (1979)

[DFL16] J. Dubois, S. Friedl and W. Lück, L^2*-Alexander torsion of 3-manifolds*, Journal of Topology, 9(3), 889–926 (2016)

[FK13] S. Friedl and T. Kim, *Taut sutured manifolds and twisted homology*, Mathematical Research Letters, 20(2), 289–303 (2013)

[FK14] S. Friedl and T. Kitayama, *The virtual fibering theorem for 3-manifolds*, Enseign. Math., 60(1–2), 79–107 (2014)

[FL18] S. Friedl and W. Lück, L^2*-Euler characteristic and the Thurston norm*, Proc. London Math. Soc., 3, 1–44 (2018)

[FL19] S. Friedl and W. Lück, *The* L^2*-torsion function and the Thurston norm of* 3*-manifolds*, Comment. Math. Helv., 94, 21-52 (2019)

[Ga83] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom., 18(3), 445–503 (1983)

[Ha62] W. Haken, *Über das Homöomorphieproblem der 3-Mannigfaltigkeiten. I.*, Mathematische Zeitschrift, Volume 80, 89–120 (1962)

[He17] G. Herrmann, *The* L^2*-Alexander torsion for Seifert fiber spaces*, Archiv der Mathematik, 109(3), 273-283 (2017)

[He18] G. Herrman, *Sutured manifolds and* L^2*-Betti numbers*, preprint (2018)

[Ja80] W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics, American Mathematical Society (1980).

[JS76] W. Jaco, P. Shalen, *Seifert fibered spaces in irreducible, sufficiently-large 3-manifolds*, Bull. Amer. Math. Soc., Volume 82(5), 765–767 (1976)

[Jo75] K. Johannson, *Łquivalences d'homotopie des varits de dimension 3*, C.R. Acad. Sci., Serie A, 281, 1009-1010 (1975)

[Ka92] O. Kakimizu, *Finding disjoint incompressible spanning surfaces for a link*, Hiroshima Math. J., 22(2), 225–236 (1992)

[Kne29] H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 38, 248–259 (1929)

[LZ06] W. Li and W. Zhang, *An* L^2*-Alexander invariant for knots*, Commun. Contemp. Math., 8, 167-187 (2006)

[Li17] Y. Liu, *Degree of* L^2*-Alexander torsion for 3-manifolds*, Invent. Math. 207, 981–1030 (2017).

[Lo92] J. Lott, *Heat kernels on covering spaces and topological invariants*, J. Differential Geom., 35(2), 471–510 (1992)

[Lü94] W. Lück, L^2*-invariants by their finite-dimensional analogues*, Geom. Funct. Anal., 4(4), 455–481, (1994)

[Lü02] W. Lück, L^2*-Invariants: Theory and Applications to Geometry and K-Theory*, Springer-Verlag Berlin Heidelberg, Survey in Math., 44 (2002)

[Lü17] W. Lück, *Twisting* L^2*-invariants with finite-dimensional representations*, Journal of Topology and Analysis, 10(4), 723-816 (2017)

[LS99] W. Lück and T. Schick, L^2*-torsion of hyperbolic manifolds of finite volume*, Geom. Funct. Anal., 9, no. 3, 518-567 (1999)

[McM02] C. McMullen, *The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology*, Ann. Sci. École Norm. Sup., 35(4), 153–171 (2002)

[Mi62a] J. Milnor, *A unique decomposition theorem for 3-manifolds*, American Journal of Mathematics, 84, 1–7 (1962)

[Mi62b] J. Milnor, *A duality theorem for Reidemeister torsion*, Ann. of Math., 76 (1), 137–147 (1962)

[Mo52] E. Moise, *Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung*, Annals of Mathematics, 56, 96–114 (1952)

[Pa57] C. Papakyriakopoulos, *On Dehn's Lemma and the asphericity of knots*, Annals of Mathematics, Volume 66, 1–26 (1957)

[Pe02] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, preprint (2002)

[Pe03] G. Perelman, *Ricci flow with surgery on three-manifolds*, preprint (2003)

[PW17] P. Przytycki and D.T. Wise, *Mixed 3-manifolds are virtually special*, J. Amer. Math. Soc., https://doi.org/10.1090/jams/886 (2017)

[Sch01] T. Schick, L^2*-determinant class and approximation of* L^2*-Betti numbers*, Trans. Amer. Math. Soc., 353, 3247–3265 (2001)

[Sei33] H. Seifert, *Topologie dreidimensionaler gefaserter Räume*, Acta Mathematica, Issue 1, Volume 60, 147–238 (1933)

[Sei35] H. Seifert, *Über das Geschlecht von Knoten*, Mathematische Annalen, 110(1), 571–592 (1935)

[Se80] J. Serre, *Trees*, (J. Stilwell), Springer-Verlag Berlin Heidelberg (1980)

[Th86a] W. Thurston, *Hyperbolic structures on 3-manifolds I: Deformation of Acylindrical manifolds*, Annals of Mathematics, No. 2, Volume 124, 203–246 (1986)

[Th86b] W. Thurston, *Hyperbolic structures on 3-manifolds II: Surface groups and 3-manifolds which fiber over the circle*, preprint (1986)

[Th86c] W. Thurston, *Hyperbolic structures on 3-manifolds III: Deformations of 3-manifolds with incompressible boundary*, preprint (1986)

[Th86] W. P. Thurston, *A norm for the homology of 3-manifolds*, Mem. Amer. Math. Soc., 59 (339) i-vi and 99–130 (1986)

[Tu01] V. Turaev, *Introduction to combinatorial torsions*, Birkhäuser Basel, Lectures in Mathematics. ETH Zürich (2001)

[Tu02] V. Turaev, *A homological estimate for the Thurston norm*, unpublished paper (2002), arXiv:math/0207267

[Wa67] F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I*, Inventiones mathematicae, 3(4), 308-333 (1967)

[Wa68] F. Waldhausen, *On Irreducible 3-Manifolds Which are Sufficiently Large*, Ann. of Math 87(1), 56–88 (1968)

Part III.

Appendix

APPENDIX A

Semi-norms on free abelian groups

We briefly recall the theory of integral semi-norms on a finitely generated free abelian group mostly because the author is not aware of a textbook style reference of this subject but it should exist. The main result (see Corollary A.12) is that the unit ball of such an integral semi-norm is the intersection of finitely many half-spaces. For the treatment we will need some standard facts from functional analysis, which we briefly recall.

A.1. Hahn-Banach and simple applications

We first recall the Hahn-Banach theorem and discuss some applications. A proof can be found in nearly every book on functional analysis e. g. the book of Hirzebruch and Scharlau [HS91]. Note that here we do not restrict to finite dimensional vector spaces.

Theorem A.1 (Hahn-Banach extension theorem)**.** *Let* N *be a semi-norm on a vector space* V*. Let* $L \subset V$ *be a linear subspace and* $f\colon L \to \mathbb{R}$ *a linear map with* $f(x) \leqslant N(x)$ *for all* $x \in L$*. Then there is a linear map* $F\colon V \to \mathbb{R}$ *with*

1. $F(x) = f(x)$ *for all* $x \in L$*,*
2. $F(x) \leqslant N(x)$ *for all* $x \in V$*.*

This theorem lets one reformulate a semi-norm in terms of linear functionals. Let V^* denote the (algebraic) dual space of V i.e. $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. We define the set

$$B^*_N := \{F \in V^* \mid F \leqslant N\}.$$

Corollary A.2. *Let* V *be a vector space and* N *be a semi-norm on* V*. For all* $v \in V$ *one has*

$$N(v) = \max_{F \in B^*_N} F(v).$$

In particular, the maximum exists.

Proof . Let $v \in V$ be arbitrary. By definition of B^* one has $N(v) \geqslant \max_{F \in B^*} F(v)$. If we set $L = \langle v \rangle$ and define $f\colon L \to \mathbb{R}, \lambda \cdot v \mapsto \lambda \cdot N(v)$, then by the Hahn-Banach extension theorem we obtain an element $F \in B^*$ with $F(v) = N(v)$. This shows the equality. □

The unit norm ball $B_N = \{v \in V \mid N(v) \leqslant 1\}$ can also be described by functionals of B_N^*. Note that every element $F \in B_N^*$ defines a half space $H_F := F^{-1}((-\infty, 1])$. Thus we have a different point of view on Corollary A.2.

Lemma A.3. *Let V be a vector space and N a semi-norm on V, then*

$$B_N = \bigcap_{F \in B_N^*} H_F.$$

Proof . Given $x \in V$ with $N(x) \leqslant 1$, one has for all $F \in B_N^*$ that $F(x) \leqslant N(x) \leqslant 1$ and hence $x \in \bigcap_{F \in B_N^*} H_F$. Conversely, if $x \in \bigcap_{F \in B_N^*} H_F$, then by Corollary A.2 we see that $N(x) = \max_{F \in B_N^*} F(x) \leqslant 1$. □

We conclude the section by answering the question under which conditions one has $N(x+y) = N(x) + N(y)$.

Proposition A.4. *Let V be a vector space and N be a semi-norm on V. Then the following statements hold:*

1. *The subset $Z := \{v \in V \mid N(v) = 0\}$ is a linear subspace. Moreover, we obtain a norm $\overline{N}$ on $W = V/Z$ by $\overline{N}([w]) := N(w)$.*
2. *For $v, w \in V$ one has $N(v+w) = N(v) + N(w)$ if and only if there exits $F \in B^*$ with $F(v) = N(v)$ and $F(w) = N(w)$.*
3. *If $N(v+w) = N(v) + N(w)$ for $v, w \in V$, then for all $s, t > 0$ one has $N(sx+ty) = sN(x) + tN(y)$.*

Proof . (1) The first part of the statement follows from the triangle inequality. For the second statement we only show that it is well-defined. The rest is left as an exercise. Let $w \in V$ and $z \in Z$ be arbitrary. We have by the triangle inequality $N(w+z) \leqslant N(w) + N(z) = N(w)$ and hence $\overline{N}$ does not depend on the choice of representative.
(2) First we assume that $N(v+w) = N(v) + N(w)$. By Corollary A.2 we obtain an $F \in B_N^*$ such that $F(x+y) = N(x+y)$. Since F is in B_N^* we have $F(x) \leqslant N(x)$ and $F(y) \leqslant N(y)$. But since $N(x+y) = N(x) + N(y)$ we also have

$$F(x) + F(y) = F(x+y) = N(x+y) = N(x) + N(y).$$

The two inequalities $F(x) \leqslant N(x)$, $F(y) \leqslant N(y)$ together with the equality $F(x) + F(y) = N(x) + N(y)$ imply $F(x) = N(x)$ and $F(y) = N(y)$.
Conversely, if we have an $F \in B_N^*$ with $F(v) = N(v)$ and $F(w) = N(w)$, then

$$N(v) + N(w) = F(v) + F(w) = F(v+w) \leqslant N(v+w) \leqslant N(v) + N(w).$$

The first inequality comes from the fact that $F \in B_N^*$ and the second one is the triangle inequality.
(3) By (2) we have an element $F \in B^*$ with $F(v) = N(v)$ and $F(w) = N(w)$ and we can do the same conclusion as before

$$sN(v) + tN(w) = F(sv+tw) \leqslant N(sv+tw) \leqslant sN(v) + tN(w).$$

□

A.2. Integral semi-norm on a free abelian group

Remark A.5. Let H be a finitely generated free abelian group. Recall that one obtains an finite dimensional $\mathbb{R}$-vector space V by defining $V \coloneqq H \otimes_{\mathbb{Z}} \mathbb{R}$. The group H sits as a lattice in V via the map $H \to V, h \mapsto h \otimes 1$. Further recall, that given a basis $B = \{b_1, \dots, b_n\}$ for H, we obtain an identification of H with $\mathbb{Z}^n$. Clearly, $\{b_1 \otimes 1, \dots b_n \otimes 1\}$ is a basis for V and we can identify the lattice H in V by the more familiar lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$.

We will state and prove everything for $\mathbb{Z}^n$ in $\mathbb{R}^n$ but the statements hold completely analogously for H and V. We chose to do it this way since in most proofs we have to pick a basis anyway and by working with $\mathbb{Z}^n$ and $\mathbb{R}^n$ the proofs look much less technical.

We consider the free abelian group $\mathbb{Z}^n$ of rank n. We call a map $N\colon \mathbb{Z}^n \to \mathbb{R}_{\geqslant 0}$ a semi-norm if for all $t \in \mathbb{Z}$ and $x, y \in \mathbb{Z}^n$ one has

1. (subadditivity) $N(x+y) \leqslant N(x) + N(y)$
2. (homogeneity) $N(tx) = |t| \cdot N(x)$.

Lemma A.6. *Let N a semi-norm on $\mathbb{Z}^n$. There is a unique extension of N to a semi-norm on the vector space $\mathbb{R}^n$.*

Proof . Let $v \in \mathbb{Q}^n$ be a rational vector and $k \in \mathbb{Z}$ such that $k \cdot v \in \mathbb{Z}^n$. We define $N(v) = \frac{N(kv)}{|k|}$. This is well-defined since if there is another $t \in \mathbb{Z}$ such that $t \cdot v \in \mathbb{Z}^n$ one has

$$\frac{N(kv)}{|k|} = \frac{|t| \cdot N(kv)}{|t| \cdot |k|} = \frac{N(tkv)}{|t| \cdot |k|} = \frac{|k| \cdot N(tv)}{|k| \cdot |t|} = \frac{N(tv)}{|t|}.$$

Next we show that this extension is subadditive and homogeneous with respect to elements in $\mathbb{Q}$. First of all let $v, w \in \mathbb{Q}^n$ be rational elements and $k, l \in \mathbb{N}$ such that $kv, lw \in \mathbb{Z}^n$. Then one has

$$N(v+w) = \frac{N(klv + klw)}{|kl|} \leqslant \frac{N(klv) + N(klw)}{|kl|} = N(v) + N(w).$$

For $p, q \in \mathbb{Z}$ with $q \neq 0$ we see

$$N\left(\frac{p}{q} \cdot v\right) = \frac{N(pkv)}{|qk|} = \frac{|p| \cdot N(kv)}{|q| \cdot |k|} = \frac{|p|}{|q|} N(v).$$

If we take a sequence of rational elements $\{v_k\}_{k \in \mathbb{N}}$ converging to some $v \in \mathbb{R}^n$, then we define

$$N(v) \coloneqq \lim_{k \to \infty} N(v_k).$$

First we show that the limit $\lim\limits_{k \to \infty} N(v_k)$ exists by showing that $\{N(v_k)\}_{k \in \mathbb{N}}$ is a Cauchy sequence. We denote by e_i the i-th standard unit vector. For every $x = (q_1, \dots, q_n)$

with $q_i \in \mathbb{Q}$ we have

$$N(x) = N\left(\sum_{i=1}^{n} q_i \cdot e_i\right) \leqslant \sum_{i=1}^{n} |q_i| N(e_i) \leqslant \max_{i\in\{1,\ldots,n\}} N(e_i) \cdot \sum_{i=1}^{n} |q_i| = \max_{i\in\{1,\ldots,n\}} N(e_i) \cdot \|x\|_1.$$

Therefore, if we set $C \coloneqq \max_{i\in\{1,\ldots,n\}} N(e_i)$, then we have

$$|N(v_k) - N(v_m)| \leqslant N(v_k - v_m) \leqslant C \cdot \|v_k - v_m\|_1$$

and hence $\{N(v_k)\}_{k\in\mathbb{N}}$ is indeed a Cauchy sequence. Again we have to check that our definition is independent of the choice of sequence. So let $\{v'_k\}_{k\in\mathbb{N}}$ be a different sequence converging against v. This implies that the sequence $\|v_k - v'_k\|_1$ converges to zero. Moreover, we have that both sequences $\{N(v_k)\}_{k\in\mathbb{N}}$ and $\{N(v'_k)\}_{k\in\mathbb{N}}$ are convergent and hence we are allowed to calculate

$$\begin{aligned} |\lim_{m\to\infty} N(v_m) - \lim_{k\to\infty} N(v'_k)| &= \lim_{m\to\infty} |N(v_m) - N(v'_m)| \leqslant \lim_{n\to\infty} |N(v_m - v'_m)| \\ &\leqslant \lim_{m\to\infty} C_N \cdot \|v_m - v'_m\|_1 = 0. \end{aligned}$$

One again easily verifies that this definition is subadditive and homogeneous over $\mathbb{R}$, which finishes the existence part of the lemma. Next we prove uniqueness.

Let $N' \colon \mathbb{R}^n \to \mathbb{R}$ be a semi-norm with $N'(x) = N(x)$ for all $x \in \mathbb{Z}^n$. Consider a rational element $v \in \mathbb{Q}^n$ with $k \cdot v \in \mathbb{Z}^n$ for some $k \in \mathbb{Z}$. Then we have

$$N'(v) = \frac{1}{|k|} \cdot N'(kv) = \frac{1}{|k|} \cdot N(kv) = N(v).$$

Note that N and N' are both continuous, which can be proven in the same way as we have proven that $\{N(v_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence. Now N and N' are continuous functions which agree on a dense subset of $\mathbb{R}^n$ and hence they are equal. □

Since the extension is unique, we will implicitly extend a semi-norm on $\mathbb{Z}^n$ to $\mathbb{R}^n$ without always mentioning it.

Definition A.7. Let N be a semi-norm on $\mathbb{Z}^n$. If N additionally satisfies

3. (integral) $N(x) \in \mathbb{N}_0$ for all $x \in \mathbb{Z}^n$

then we call N an *integral* semi-norm.

The first observation for an integral semi-norm is that it is easier to see when it extend to a norm on $\mathbb{R}^n$.

Lemma A.8. *If* N *is an integral semi-norm on* $\mathbb{Z}^n$ *with* $N(x) > 0$ *for all* $x \in \mathbb{Z}^n \setminus \{0\}$*, then* $N(y) > 0$ *for all* $y \in \mathbb{R}^n \setminus \{0\}$*.*

***Proof*.** First we recall Dirichlet's theorem [HW08, Theorem 15]. For any vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and any $\epsilon > 0$ there exists a vector $k \in \mathbb{Z}^n$ and $q \in \mathbb{Z}$, such that $\|q \cdot \alpha - k\|_1 < \epsilon$. Let $\alpha \in \mathbb{R}^n$ be arbitrary. Let e_i be the i-th unit vector and $C := \max_{i \in \{1,\dots,n\}} N(e_i)$. We set $\epsilon = \frac{1}{2 \cdot C}$ and apply the theorem of Dirichlet to obtain a $q \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ such that $\|q \cdot \alpha - k\|_1 < \epsilon$. We calculate

$$|N(q \cdot \alpha) - N(k)| \leqslant |N(q \cdot \alpha - k)| \leqslant C \cdot \|q \cdot \alpha - k\|_1 < C \cdot \epsilon = 1/2$$

Since N was integral we have $N(k) \geqslant 1$ and hence $N(q \cdot \alpha) \geqslant 1/2$ which implies $N(\alpha) > 0$. □

Remark A.9. The assumption on being an integral semi-norm is necessary. We can define the semi norm $f \colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = |\sqrt{2}x + y|$. It satisfies the assumption that $f(x, y) > 0$ for all $(x, y) \in \mathbb{Z}^2 \setminus \{0\}$ but clearly one has $f(-1, \sqrt{2}) = 0$.

Lemma A.10. *If N is an integral semi-norm on $\mathbb{Z}^n$, then for all $a, b \in \mathbb{Z}^n$ there exists an $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ we have*

$$N(a + mb + k \cdot b) = N(a + mb) + k \cdot N(b).$$

***Proof*.** One considers the function $\varphi \colon \mathbb{Z} \to \mathbb{Z}$, $n \mapsto N(a + (n + 1)b) - N(a + nb)$. Since N is convex, one has $N(a + (n + 1)b) + N(a + (n - 1)b) \geqslant 2N(a + nb)$. This is equivalent to

$$N(a + (n + 1)b) - N(a + nb) \geqslant N(a + nb) - N(a + (n - 1)b).$$

Hence we have $\varphi(n + 1) \geqslant \varphi(n)$ or with other words φ is monotonously increasing. By the inverted triangle inequality one has $\varphi(n) = N(a+(n+1)b)-N(a+nb) \leqslant N(b)$ or with other words φ is bounded. Therefore, there exists an $m \in \mathbb{N}$ such that for all $n \geqslant m$ we have $c := \varphi(m) = \varphi(n)$. Using a telescope sum argument one deduces:

$$N(a + mb + kb) - N(a + mb) = \sum_{i=0}^{k} \varphi(m + i) = kc$$

And we conclude the lemma by $c = \lim_{k \to \infty} \frac{N(a+mb+kb)-N(a+mb)}{k} = N(b)$. □

Now we want to combine the lemma with the theorem of Hahn-Banach to prove that the unit ball of an integral semi-norm is a finite sided polyhedra. First we introduce some notation. Let $\langle \ , \ \rangle$ be the standard scalar product on $\mathbb{R}^n$. Note that this gives an identification of $\mathbb{Z}^n$ with $(\mathbb{Z}^n)^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z})$ and of $\mathbb{R}^n$ with $\mathbb{R}^{n*} = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$. If N is a semi-norm we write B^*_N for the set

$$B^*_N = \{y \in \mathbb{R}^n \mid \forall_{x \in \mathbb{R}^n} \langle x, y\rangle \leqslant N(x)\}$$

and

$$B^*_{N,\mathbb{Z}} = \{x \in \mathbb{Z}^n \mid \forall_{x \in \mathbb{Z}^n} \langle x, y\rangle \leqslant N(x)\}.$$

If N is clear from the context, then we drop it from the notation. We make two observations, which we leave as elementary exercises. The first observation is that $B^*_{N,\mathbb{Z}}$ is a subset of B^*_N i. e. if for all $x \in \mathbb{Z}^n$ one has $\langle x, y\rangle \leqslant N(x)$, then the same holds for all $x \in \mathbb{R}^n$. The second observation is that B^* is compact and hence $B^*_{\mathbb{Z}}$ is finite.

The next proposition should be compared to Corollary A.2, but here we need that we work over a finite dimensional vector space.

Proposition A.11. *Let N be an integral semi-norm on $\mathbb{R}^n$. One has for all $x \in \mathbb{R}^n$ that*

$$\max_{y\in B^*_{\mathbb{Z}}} \langle y, x\rangle = N(x).$$

Proof . For a simpler notation we assume that $n = 3$. We start with a primitive element $x \in \mathbb{Z}^3$. We extend x by $b_1, b_2 \in \mathbb{Z}^3$ to a basis of $\mathbb{Z}^3$. By Lemma A.10 there is a $k_1 \in \mathbb{N}$ such that

$$N(x + b_1 + k_1x) = N(x) + N(b_1 + k_1x).$$

We write $b'_1 = b_1 + k_1x$. By applying the previous lemma again we also obtain a $k_2 \in \mathbb{N}$ with the property that

$$N(x + b'_1 + b_2 + k_2 \cdot (x + b'_1)) = N(x + b'_1) + N(b_2 + k_2 \cdot (x + b'_1)).$$

If we further set $b'_2 = b_2 + k_2 \cdot (x + b'_1)$, then the equation becomes

$$N(x + b'_1 + b'_2) = N(x + b'_1) + N(b'_2) = N(x) + N(b'_1) + N(b'_2).$$

We stop here for notational convenience. It should be clear how to generalise this construction to higher rank. The important observation is that $\{x, b'_1, b'_2\}$ is still a basis of $\mathbb{Z}^3$. Therefore, we define the linear map

$$\begin{aligned} F\colon \mathbb{Z}^3 &\longrightarrow \mathbb{Z} \\ \alpha x + \beta b'_1 + \gamma b'_2 &\longmapsto \alpha N(x) + \beta N(b'_1) + \gamma N(b'_2). \end{aligned}$$

We clearly have $F(x) = N(x)$. We are left to show that $F \in B^*_{\mathbb{Z}}$. Let $\alpha, \beta, \gamma \in \mathbb{Z}$ be positive. It follows from Proposition A.4 (3) that

$$N(\alpha x + \beta b'_1 + \gamma b'_2) = \alpha N(x) + \beta N(b'_1) + \gamma N(b'_2)$$

and hence $N(\alpha x + \beta b'_1 + \gamma b'_2) = F(\alpha x + \beta b'_1 + \gamma b'_2)$. For the case that at least one of the coefficients is negative e. g. the coefficient of b_1 we consider the inequality

$$F(\alpha x - \beta b'_1 + \gamma b'_2) = N(\alpha x + \gamma b'_2) - \beta N(b'_1) \leqslant N(\alpha x - \beta b'_1 + \gamma b'_2).$$

The other cases are analogous and we conclude $F \in B^*_{\mathbb{Z}}$.

We proved the statement for all primitive $x \in \mathbb{Z}^3$. Now if $k \in \mathbb{Q}$ with $k > 0$ we have for a primitive $x \in \mathbb{Z}^3$ that

$$N(k \cdot x) = k \cdot N(x) = k \cdot \max_{y\in B^*_{\mathbb{Z}}} \langle y, x\rangle = \max_{y\in B^*_{\mathbb{Z}}} \langle y, kx\rangle \tag{A.1}$$

which proves the statement for all vectors in $\mathbb{Q}^3$. Since both sides are continuous in x it follows for all $x \in \mathbb{R}^3$. $\square$

Corollary A.12. *Let* N *be a integral semi-norm on* $\mathbb{R}^n$*. The unit ball* B_N *is the intersection of finitely many half spaces.*

Proof . This is the same as Lemma A.3. For $F \in B^*$ we write $H_F = F^{-1}((-\infty, 1])$ and we have

$$B_N = \bigcap_{F \in B^*_{\mathbb{Z}}} H_F.$$

As mentioned before $B^*_{\mathbb{Z}}$ consist of finitely many points. $\square$

APPENDIX B

Twisted coefficients and Poincaré duality

B.1. Definition and basic properties

We start with a fairly easy observation. If $\widehat{X}$ is a topological space (not necessary connected) on which a group G acts freely and properly from left, then the action induces a free $\mathbb{Z}[G]$-module structure on $C_*(\widehat{X};\mathbb{Z})$. Since the boundary operator is natural and equivariant, it is a chain complex over $\mathbb{Z}[G]$.

The main example of a topological space with such an action comes from universal covers. We fix some notation for the rest of this chapter. Let M be a *connected* manifold and denote by $\pi := \pi_1(M, x_0)$ the fundamental group. Let $p\colon \widetilde{M} \to M$ be the universal cover and $A \subset M$ a subset. Then π acts on $\widetilde{M}$ via deck transformation and $\widetilde{A} := p^{-1}(A)$ is a π-invariant subset. If E_M is a *right* $\mathbb{Z}[\pi]$-module, then we define (co)homology of the pair (M, A) with values in E_M by the homology of the following chain complexes:

$$\begin{aligned}
C_*(M, A; E_M) &:= E_M \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M}, \widetilde{A}; \mathbb{Z}),\\
C^*(M, A; E_M) &:= \operatorname{Hom}_{\mathbb{Z}[\pi]}\left(\overline{C_*(\widetilde{M}, \widetilde{A}; \mathbb{Z})}, E_M\right),
\end{aligned}$$

where the notation $\overline{C_*(\widetilde{M}, \widetilde{A}; \mathbb{Z})}$ means that we endow the abelian group $C_*(\widetilde{M}, \widetilde{A}; \mathbb{Z})$ with a *right* $\mathbb{Z}[\pi]$-module structure by $\sigma * \gamma := \gamma^{-1} \cdot \sigma$ for all $\sigma \in C_*(\widetilde{M}, \widetilde{A}; \mathbb{Z})$ and $\gamma \in \pi$.

Moreover, if we have a subset $X \subset M$ (which is not necessarily connected), then we define $\widetilde{X} := p^{-1}(X)$ and consider the (co)homology of X with respect to the coefficient system coming from M by

$$\begin{aligned}
C_*(X; E_M) &:= E_M \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}; \mathbb{Z}),\\
C^*(X; E_M) &:= \operatorname{Hom}_{\mathbb{Z}[\pi]}\left(\overline{C_*(\widetilde{X}; \mathbb{Z})}, E_M\right),
\end{aligned}$$

with generalisation to pairs (X, Y) in M by

$$\begin{aligned}
C_*(X, Y; E_M) &:= E_M \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z}),\\
C^*(X, Y; E_M) &:= \operatorname{Hom}_{\mathbb{Z}[\pi]}\left(\overline{C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})}, E_M\right).
\end{aligned}$$

We summarise the basic properties of twisted coefficients in the following theorem, which should be compared to the untwisted case.

Theorem B.1. *Let M be a connected manifold with fundamental group π, and E_M a right $\mathbb{Z}[\pi]$-module.*

1. *Given $Y \subset X \subset M$, there is a long exact sequence of pairs in homology*

$$\dots \to H_k(Y; E_M) \to H_k(X; E_M) \to H_k(X, Y; E_M) \to H_{k-1}(Y; E_M) \to \dots$$

and cohomology

$$\dots \to H^k(X, Y; E_M) \to H^k(X; E_M) \to H^k(Y; E_M) \to H^{k+1}(X, Y; E_M) \to \dots .$$

2. *Suppose we have a chain of subspaces $Z \subset Y \subset X \subset M$ and the closure of Z is contained in the interior of Y, then the inclusion $(X \setminus Z, Y \setminus Z) \to (X, Y)$ induces an isomorphism in homology and cohomology i. e.*

$$H_k(X\backslash Z, Y\backslash Z; E_M) \xrightarrow{\sim} H_k(X, Y; E_M) \quad \textit{and} \quad H^k(X\backslash Z, Y\backslash Z; E_M) \xleftarrow{\sim} H^k(X, Y; E_M).$$

3. *If $U_1 \subset U_2 \subset M$ and $V_1 \subset V_2 \subset M$ are open subsets in M, then there is a long exact sequence in homology*

$$\begin{tikzcd} \dots \arrow[r] & H_k(U_1 \cap V_1, U_2 \cap V_2; E_M) \arrow[r] & \begin{matrix} H_k(U_1, U_2; E_M) \\ \oplus \\ H_k(V_1, V_2; E_M) \end{matrix} \arrow[r] & H_k(U_1 \cup V_2, U_2 \cup V_2; E_M) \arrow[dll] \\ & H_{k-1}(U_1 \cap V_1, U_2 \cap V_2; E_M) \arrow[r] & \dots \end{tikzcd}$$

and cohomology

$$\begin{tikzcd} \dots \arrow[r] & H_k(U_1 \cup V_1, U_2 \cup V_2; E_M) \arrow[r] & \begin{matrix} H_k(U_1, U_2; E_M) \\ \oplus \\ H_k(V_1, V_2; E_M) \end{matrix} \arrow[r] & H_k(U_1 \cap V_2, U_2 \cap V_2; E_M) \arrow[dll] \\ & H_{k-1}(U_1 \cup V_1, U_2 \cup V_2; E_M) \arrow[r] & \dots . \end{tikzcd}$$

4. *If the inclusion $Y \to X$ is a homotopy equivalence, then the inclusion induces the isomorphisms:*

$$H_k(Y; E_M) \xrightarrow{\sim} H_k(X; E_M) \quad \textit{and} \quad H^k(X; E_M) \xleftarrow{\sim} H^k(Y; E_M).$$

5. *If* $U_1 \subset U_2 \subset \dots$ *is a sequence of open sets in* M *with* $U = \bigcup_{i \in \mathbb{N}} U_i$*, then the inclusions induce an isomorphism*

$$\varinjlim_{i \in \mathbb{N}} C_*(U_i; E_M) = C_*(U; E_M).$$

Since the proofs are essentially the same as in the classical case, we will only sketch the arguments and focus on what is different. We also warn the reader that we give the wrong proof of statement (4). This is due to the fact, that we developed the theory of twisted coefficients only for inclusions and hence a homotopy inverse does not fit in our theory. Therefore, statement (4) will be deduced from the following two elementary lemmas.

Lemma B.2 (The Covering Homotopy Theorem)**.** *[Br93, Chapter III Theorem 3.4+remark after proof] Given a covering* $p\colon \widetilde{X} \to X$*, a homotopy* $H\colon Y \times I \to X$*, and a lift* $\widetilde{h}\colon Y \to \widetilde{X}$ *of* $H(-,0)$*, there exists a unique lift* $\widetilde{H}\colon Y \times I \to \widetilde{X}$ *of* H *with* $\widetilde{h} = \widetilde{H}(-,0)$*.*

Lemma B.3. *If* $f_*\colon C_* \to D_*$ *is a quasi-isomorphism of bounded free* $\mathbb{Z}[\pi]$*-modules, then* f_* *is a chain homotopy equivalence.*

Proof . Since f is a quasi-isomorphism we know that the $\mathrm{cone}(f)_*$ is acyclic. By assumption C_* and D_* are free $\mathbb{Z}[\pi]$-modules and so is $\mathrm{cone}(f)_*$. But this guarantees the existence of a chain homotopy $\mathrm{Id}_{\mathrm{cone}\, f_*} \simeq_P 0$, since we can view $\mathrm{cone}(f)_*$ as a free resolution of 0 and any two such resolutions are chain homotopic. Recall that chain homotopy means

$$\partial^{\mathrm{cone}\, f_*} \circ P + P \circ \partial^{\mathrm{cone}\, f_*} = \mathrm{Id}_{\mathrm{cone}\, f_*} \tag{B.1}$$

If we write P as a matrix

$$P_n = \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} : \begin{matrix} C_{n-1} \\ D_n \end{matrix} \to \begin{matrix} C_n \\ D_{n+1} \end{matrix},$$

then one easily verifies using Equation (B.1), that $P_*^{12}\colon D_* \to C_*$ is a chain homotopy inverse of f_*, where the chain homotopies are given by P_*^{11} and P_*^{22}. □

Proof of Theorem B.1 . Recall that $p\colon \widetilde{M} \to M$ denotes the universal cover and for any subspace $X \subset M$ we write $\widetilde{X} = p^{-1}(X)$.

For statement (1) we consider the short exact sequence $0 \to C_*(\widetilde{Y}; \mathbb{Z}) \to C_*(\widetilde{X}; \mathbb{Z}) \to C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z}) \to 0$ of *free* $\mathbb{Z}[\pi]$-modules. Since the modules are free, the sequence stays exact after applying the functors $E_M \otimes_{\mathbb{Z}[\pi]} -$ and $\mathrm{Hom}_{\mathbb{Z}[\pi]}(-, E_M)$.

Recall the proof of statement (2) and (3) in the classical case as it is done for example in Bredon's book [Br93, Chapter IV Section 17]. The main ingredient is to show that the inclusion of chain complexes $C_*^{\mathcal{U}}(X;\mathbb{Z}) \to C^*(X;\mathbb{Z})$ induces an isomorphism on homology [Br93, Theorem 17.7]. Here $\mathcal{U}$ is an open cover of X and $C_*^{\mathcal{U}}(X;\mathbb{Z})$ is the free abelian group generated by simplexes σ for which there is a $U \in \mathcal{U}$ such that $\sigma\colon \Delta^* \to U$. This is done by defining the barycentric subdivision

$\Upsilon_*\colon C_*(X;\mathbb{Z}) \to C_*(X;\mathbb{Z})$ and a chain homotopy T between Υ_* and the identity [Br93, Lemma 17.1]. The important thing for us to observe is that both maps are natural [Br93, Claim (1) in proof of Lemma 17.1]. Hence for a twisted chain $\Upsilon(e \otimes_{\mathbb{Z}[\pi]} \sigma) := e \otimes_{\mathbb{Z}[\pi]} \Upsilon(\sigma)$ is well-defined, because

$$\begin{aligned}
\Upsilon(e \otimes_{\mathbb{Z}[\pi]} \gamma\sigma) &= e \otimes_{\mathbb{Z}[\pi]} \Upsilon(\gamma\sigma) \\
&= e \otimes_{\mathbb{Z}[\pi]} \gamma\Upsilon(\sigma) \quad \text{(naturality of } \Upsilon) \\
&= e\gamma \otimes_{\mathbb{Z}[\pi]} \Upsilon(\sigma) = \Upsilon(e\gamma \otimes_{\mathbb{Z}[\pi]} \sigma).
\end{aligned}$$

The same holds for T and from now on one can follow the classical proofs. Alternatively, one could invoke Lemma B.3.

Next we prove statement (4). Let $f\colon X \to Y \subset X$ be a homotopy inverse of the inclusion and $H\colon X \times I \to X$ a homotopy between Id_X and f. Since $p\colon \widetilde{X} \to X$ is a covering and $\mathrm{Id}_{\widetilde{X}}$ is a lift of $H(p(-), 0)$, we get by Lemma B.2 a lift $\widetilde{H}\colon \widetilde{X} \times I \to \widetilde{X}$ of the homotopy H. One easily verifies now that the inclusion $\widetilde{Y} \to \widetilde{X}$ induces a homotopy equivalence where a homotopy inverse is given by $\widetilde{H}(-, 1)$. Hence the inclusion induces an quasi-isomorphism $C_*(\widetilde{Y};\mathbb{Z}) \to C_*(\widetilde{X};\mathbb{Z})$ and the claim follows from Lemma B.3.

Statement (5) is completely the same proof as in the classical case. □

B.2. Poincaré duality

In this section we prove Poincaré duality with twisted coefficients. We follow very closely the proof of Poincaré duality as done in Bredon's book [Br93, Chapter VI Section 8]. The logic of his proof is unchanged, but some arguments and definitions have to be adjust to the twisted setting.

Preliminaries for Poincaré duality

Note that we can view $\mathbb{Z}$ as a $\mathbb{Z}[\pi]$-module with trivial π-action, which we will denote by $\mathbb{Z}^{\mathrm{triv}}$. We have the following useful lemma.

Lemma B.4. *Given any subset $X \subset M$, we have canonical isomorphisms between $C_*(X;\mathbb{Z})$ and $C_*(X;\mathbb{Z}^{\mathrm{triv}}_M)$ and between $C^*(X;\mathbb{Z})$ and $C^*(X;\mathbb{Z}^{\mathrm{triv}}_M)$, where $C_*(X;\mathbb{Z})$ and $C^*(X;\mathbb{Z})$ means the untwisted singular chain complexes.*

Proof . The isomorphism is given by lifting a simplex, which is always possible since a simplex is simply connected. If one has two different choices of lifts, then they differ by an element in π. But the action of $\mathbb{Z}[\pi]$ on $\mathbb{Z}$ is trivial and hence this indeterminacy vanishes. □

In contrast to most other authors we will keep the notational difference between $C_*(X;\mathbb{Z})$ and $C_*(X;\mathbb{Z}^{\mathrm{triv}}_M)$ to emphasise were our simplexes live.

In order to define the cap product we need one more notation. Given an n-simplex σ, we define the p-simplexes $\sigma\lfloor_p$ and $\sigma\rfloor_p$ by

$$\sigma\rfloor_p(t_0,\dots,t_p) := \sigma(t_0,\dots,t_p,0,\dots,0),$$
$$\sigma\lfloor_p(t_0,\dots,t_p) := \sigma(0,\dots,0,t_0,\dots,t_p).$$

Now the cap product is defined by the following map.

Definition B.5 (Cap product)**.**

$$\begin{aligned}\frown\colon C^p(X;E_M)\times C_k(X,Y;\mathbb{Z}_M^{\mathrm{triv}}) &\longrightarrow C_{k-p}(X,Y;E_M)\\ (\psi, n\otimes_{\mathbb{Z}[\pi]}\sigma) &\longmapsto n\cdot\psi(\sigma\lfloor_p)\otimes_{\mathbb{Z}[\pi]}\sigma\rfloor_{k-p}.\end{aligned}$$

One easily checks that this is well-defined, since

$$\begin{aligned}\psi\frown(n\otimes_{\mathbb{Z}[\pi]}\gamma\sigma) &= n\cdot\psi(\gamma\sigma\lfloor_p)\otimes_{\mathbb{Z}[\pi]}\gamma\sigma\rfloor_{k-p}\\ &= n\cdot\psi(\sigma\lfloor_p)\gamma^{-1}\otimes_{\mathbb{Z}[\pi]}\gamma\sigma\rfloor_{k-p} = \psi\frown(n\otimes_{\mathbb{Z}[\pi]}\sigma).\end{aligned}$$

The classical formula together with the fact that ∂ and δ are natural shows the formula (up to signs coming from sign conventions):

$$\partial(f\frown c) = \delta(f)\frown c + (-1)^k f\frown\partial c. \tag{B.2}$$

Therefore, the cap product descends to a map in twisted (co)homology.

Let $K\subset M$ be a compact subset of M. We define the (twisted) Čech cohomology groups

$$\check{H}^p(K;E_M) := \varinjlim_{K\subset U\subset M} H^p(U;E_M),$$

where the direct limit runs over all open sets in M containing K. Since cohomology is contravariant, we define the order on open sets in the reversed way i.e. $U\leqslant V$ if $V\subset U$.

Remark B.6. In later applications we would like to work with singular cohomology rather than Čech cohomology. Hence we mention the following trivial observation. Suppose that $K\subset M$ admits a family of open neighbourhoods $U_1\supset U_2\supset\dots$ with the following two properties:

1. for every $i\in\mathbb{N}$ the inclusion $K\to U_i$ is a homotopy equivalence,
2. for every open neighbourhood V of K there is a $i\in\mathbb{N}$ such that $U_i\subset V$,

then the canonical map $\check{H}^p(K;E_M)\to H^p(K;E_M)$ is an isomorphism. Such a family exists for example if K is a smooth submanifold of M. We refer to Bredon's book [Br93, Corollary E.6] for a much stronger version of this remark. It turns out to be sufficient that K is homotopy equivalent to a CW-complex in order that the map $\check{H}^p(K;E_M)\to H^p(K;E_M)$ is an isomorphism.

Now we assume that M is oriented. Being oriented gives us for any closed subset $A \subset M$ a preferred element $\theta_A \in H_n(M, M \setminus A; \mathbb{Z}^{\mathrm{triv}}) \cong H_n(M, M \setminus A; \mathbb{Z})$ (which restricts for all $x \in A$ to the generator in $H_n(M, M \setminus \{x\}; \mathbb{Z}^{\mathrm{triv}})$).

For any open set $U \subset M$ containing K let

$$\mathrm{ex}_U \colon H_n(M, M \setminus K; \mathbb{Z}^{\mathrm{triv}}) \to H_n(U, U \setminus K; \mathbb{Z}^{\mathrm{triv}})$$

be the inverse of the inclusion given by the excision isomorphism i. e. if $j \colon U \to M$ is the inclusion, then $j_* \circ \mathrm{ex}_U = \mathrm{Id}$. We obtain a map

$$\begin{aligned} D_U \colon H^p(U; E_M) &\longrightarrow H_n(M, M \setminus K; E_M) \\ \phi &\longmapsto j_*(\phi \frown \mathrm{ex}_U(\theta_K)). \end{aligned}$$

Given another open set $V \subset U$, we denote by $i \colon V \to U$ the inclusion. One easily calculates:

$$D_V(i^*\phi) = j_* i_*(i^*\phi \frown \mathrm{ex}_V(\theta_K)) = j_*(\phi \frown i_* \mathrm{ex}_V(\theta_K)) = j_*(\phi \frown \mathrm{ex}_U(\theta_K)) = D_U(\phi)$$

or with other words the following diagram commutes:

$$\begin{tikzcd} H^p(U; E_M) \arrow[dr, "D_U"] \arrow[d, "i^*"'] & \\ H^p(V; E_M). \arrow[r, "D_V"] & H_{n-p}(M, M \setminus K; E_M) \end{tikzcd}$$

By the universal property of the direct limit we obtain the dualising map

$$D_K \colon \check{H}^p(K; E_M) \to H_{n-p}(M, M \setminus K; E_M).$$

We can finally state the theorem which to prove is the purpose of this chapter.

Theorem B.7 (Poincaré duality)**.** *The map* $D_K \colon \check{H}^p(K; E_M) \to H_{n-p}(M, M \setminus K; E_M)$ *is an isomorphism for all compact subsets* $K \subset M$.

The proof will be an application of the following lemma.

Lemma B.8 (Bootstrap lemma)**.** *[Br93, Chapter VI Lemma 7.9] Let* $P_M(K)$ *be a statement about compact sets* K *in* M. *If* $P_M(\cdot)$ *satisfies the following three conditions:*

1. $P_M(K)$ *holds true for all compact subsets* $K \subset M$ *with the property that for all* $x \in K$ *the inclusions* $\{x\} \to K$ *and* $M \setminus K \to M \setminus \{x\}$ *are deformation retracts,*

2. *if* $P_M(K_1), P_M(K_2)$ *and* $P_M(K_1 \cap K_2)$ *is true, then* $P_M(K_1 \cup K_2)$ *is true,*

3. *if* $\ldots \subset K_2 \subset K_1$ *and* $P_M(K_i)$ *is true for all* $i \in \mathbb{N}$, *then* $P_M(\bigcap_{i \in \mathbb{N}} K_i)$ *is true,*

then $P_M(K)$ *is true for all* $K \subset M$.

It turns out that condition (3) is the easiest to verify. It follows from formal properties about direct limits. For the verification of condition (1) we have do to one explicit calculation. This is the content of the next lemma.

Lemma B.9. *Let $x \in M$ be a point. The map $D_{\{x\}}\colon \check{H}^0(\{x\}; E_M) \to H_n(M, M \setminus \{x\}; E_M)$ is an isomorphism.*

Proof . Let $p\colon \widetilde{M} \to M$ be the universal cover. Since x is a point in a manifold we can calculate the dualising map $D_{\{x\}}$ by taking the limit over open neighbourhoods U of x with the following two properties:

1. U is contractible,

2. for any connected component $\overline{U} \subset p^{-1}(U)$ the map $p|_{\overline{U}}$ is a homeomorphism.

This can be done, since any neighbourhood of x contains a neighbourhood with these two properties. Let U be such a neighbourhood of x and $\overline{U} \subset p^{-1}(U)$ a fixed connected component. This choice of connected component gives as an isomorphism $H^0(U; E_M) \cong E$ as follows. Let $f \in H^0(U; E_M)$ be arbitrary and $\overline{x} \in \overline{U}$ be a point in our connected component, then we get an element in E_M by evaluating $f([\overline{x}])$. Conversely, given an element $e \in E_M$, we can construct a function in $H^0(U; E_M)$ by setting $f([\overline{x}]) = e$ for all $\overline{x} \in \overline{U}$. Note that there is a unique way to extend f equivariantly to $C_0(p^{-1}(U); \mathbb{Z})$.

Next we are going to construct a representative of the orientation class $\theta_K \in H_n(M, M \setminus \{x\}; \mathbb{Z}^{\text{triv}})$ for which it is very simple to calculate the dualising map. Let $\overline{x}$ be the pre-image of x in $\overline{U}$. We take a cycle $\sum_{i=1}^{d} k_i \sigma_i$ which generates $H_n(\overline{U}, \overline{U} \setminus \{\overline{x}\}; \mathbb{Z})$. By excision and Lemma B.4 one sees that $1 \otimes_{\mathbb{Z}[\pi]} \sum_{i=1}^{d} k_i \sigma_i$ is a generator of $H_n(M, M \setminus \{x\}; \mathbb{Z}^{\text{triv}})$.

Using the isomorphism $H^0(U; E_M) \cong E$ from above the dualsing map becomes $D_{\{x\}}\colon E \to H_n(M, M\setminus\{x\}; E_M), e \mapsto e \otimes_{\mathbb{Z}[\pi]} \sum_{i=1}^{d} k_i \sigma_i$. This is clearly an isomorphism, since on the chain level we have:

$$C_*(U, U \setminus \{x\}; E_M) = E_M \otimes_{\mathbb{Z}[\pi]} \bigoplus_{\gamma \in \pi} C_*(\gamma\overline{U}, \gamma\overline{U} \setminus \{\gamma\overline{x}\}; \mathbb{Z}) \cong E_M \otimes_{\mathbb{Z}} C_*(\overline{U}, \overline{U} \setminus \{\overline{x}\}; \mathbb{Z}).$$

□

In order to verify condition (2) of the bootstrap lemma we will need the following lemma (compare [Br93, Lemma 8.2]).

Lemma B.10. *If K and L are two compact subsets of an oriented connected manifold M with orientation θ, then the diagram*

$$\begin{tikzcd}
\dots \arrow[r] & \check{H}^p(K \cup L; E_M) \arrow[r] \arrow[d, "D_{K\cup L}"] & \begin{array}{c}\check{H}^p(K; E_M)\\ \oplus\\ \check{H}^p(L; E_M)\end{array} \arrow[r] \arrow[d, "D_K \oplus D_L"] & \check{H}^p(K \cap L; E_M) \arrow[r] \arrow[d, "D_{K\cap L}"] & \check{H}^{p+1}(K \cup L; E_M) \arrow[r] \arrow[d, "D_{K\cup L}"] & \dots \\
\dots \arrow[r] & H_{n-p}(M, M \setminus (L \cup K); E_M) \arrow[r] & \begin{array}{c}H_{n-p}(M, M \setminus K; E_M)\\ \oplus\\ H_{n-p}(M, M \setminus L; E_M)\end{array} \arrow[r] & H_{n-p}(M, M \setminus (L \cap K); E_M) \arrow[r] & H_{n-p-1}(M, M \setminus (L \cup K); E_M) \arrow[r] & \dots
\end{tikzcd}$$

commutes and has exact rows.

Proof . The rows are exact by Mayer-Vietoris and the fact that direct limit is an exact functor. The commutativity of the squares are clear except for the last one involving the boundary map. This will be a painful diagram chase. Let $U \supset K$ and $V \supset L$ be open neighbourhoods containing K resp. L. The sequence in the top row comes from the short exact sequence ($\mathcal{U} = \{U, V\}$):

$$0 \longrightarrow C^*_{\mathcal{U}}(U \cup V; E_M) \longrightarrow C^*(U; E_M) \oplus C^*(V; E_M) \longrightarrow C^*(U \cap V; E_M) \longrightarrow 0$$

Note that an element $\phi \in \check{H}^p(K \cap L; E_M)$ will already be represented by same element $f \in C^p(U \cap V; E_M)$ for some U and V as above. We can extend f to an element $\overline{f} \in C^p(M; E_M)$ by

$$\overline{f}(\sigma) = \begin{cases} f(\sigma) & \text{if } \operatorname{Im}\sigma \subset \widetilde{U} \cap \widetilde{V}, \\ 0 & \text{else.} \end{cases}$$

Note that $\overline{f} \in C^p(M; E_M)$, since $\widetilde{U} \cap \widetilde{V} = p^{-1}(U \cap V)$ is an equivariant subspace and hence $\overline{f}$ is equivariant. If we consider $\overline{f}$ as an element in $C^p(U; E_M)$, then the cocycle $\delta(\phi)$ is represented by the function $h \in C^{p+1}(U \cup V; E_M)$ with

$$h(\sigma) = \begin{cases} \delta(\overline{f})(\sigma) & \text{if } \operatorname{Im}\sigma \subset \widetilde{U}, \\ 0 & \text{else.} \end{cases}$$

Since ϕ is a cocycle, we have $\delta(f)(\sigma) = 0$ for $\sigma \in C_*(\widetilde{U} \cap \widetilde{V}; \mathbb{Z})$ and hence we stress out that if σ is a simplex, whose image is completely contained in $\widetilde{V}$ then $h(\sigma) = 0$. We can represent our orientation class $\theta \in H_n(M, M \setminus (K \cup L))$ by a cycle

$$\begin{aligned} a = b + c + d + e \quad \text{with} \quad & b \in C_n(U \cap V; \mathbb{Z}^{\text{triv}}) \quad c \in C_n(U \setminus (U \cap L); \mathbb{Z}^{\text{triv}}) \\ & d \in C_n(V \setminus (V \cap K); \mathbb{Z}^{\text{triv}}), \quad e \in C_n(M \setminus (K \cup L); \mathbb{Z}^{\text{triv}}) \end{aligned}$$

Obviously e does not play a role, since we kill it in the end. With these representatives one computes that $\delta(\phi)(\theta)$ is represented by

$$h \frown (b + c + d) = \delta(\overline{f}) \cap c + h \frown d + \delta(f) \cap b = \delta(\overline{f}) \cap c.$$

The pairing of h with b is zero, since f was a cocycle in $C^*(U \cap V; E_M)$. The pairing of h with d is zero, since d consist of simplexes with image in $\widetilde{V}$.

The lower sequence comes from the short exact sequence:

$$0 \rightarrow C_*(M, M \setminus (K \cup L); E_M) \rightarrow \begin{matrix} C_*(M, M \setminus K; E_M) \\ \oplus \\ C_*(M, M \setminus L; E_M) \end{matrix} \rightarrow C_*(M, M \setminus (K \cap L); E_M) \rightarrow 0.$$

Before we compute the other side $\partial(\phi \frown ex_{U\cap V}(\theta))$, we recall that the cap product is natural on the chain complex level i. e. the following diagram commutes:

$$\begin{tikzcd}
C^p(U;E_M) \times C_n(U, U\setminus K; \mathbb{Z}_M^{\mathrm{triv}}) \arrow[r] & C_*(U, U\setminus K; E_M) \arrow[d] \\
C^p(M;E_M) \times C_n(M, M\setminus K; \mathbb{Z}_M^{\mathrm{triv}}) \arrow[u] \arrow[r] & C_*(M, M\setminus K; E_M).
\end{tikzcd}$$

Therefore, we use the representatives from above. To construct the boundary map ∂, we take as the pre-image of $\bar{f} \frown a \in C_*(M, M\setminus(K\cap L); E_M)$ the element $(\bar{f} \frown a, 0) \in C_*(M, M\setminus K; E_M) \oplus C_*(M, M\setminus L; E_M)$. Then one computes in $C_*(M, M\setminus K; E_M)$

$$\begin{aligned}
\partial(\bar{f} \frown a) &= \delta(\bar{f}) \frown a \pm f \frown \partial a \\
&= \delta(\bar{f}) \frown a \quad (\text{since } f \frown \partial a \in C_{n-p-1}(M \setminus (K\cup L); E_M)) \\
&= \delta(\bar{f}) \frown b + c + d + e \\
&= \delta(\bar{f}) \frown (c + d) \quad (\text{same reason as above}) \\
&= \delta(\bar{f}) \frown c \quad (\text{since } d \in C_{n-p}(V \setminus (K\cap V); E_M)).
\end{aligned}$$

Thus the element $\partial(\phi \frown ex_{U\cup V}(\theta))$ is represented by $\delta(\bar{f}) \frown c$. □

Proof of Theorem B.7 . Let $P_M(K)$ be the statement that the map D_K is an isomorphism. It is sufficient to verify condition (1), (2), and (3) of the bootstrap lemma. We start with verifying (1). In the case that $K = \{x\}$ is just a point we have already seen in Lemma B.9 that the statement holds true. For a general compact K with the property of (1) the statement follows from the following commutative diagram:

$$\begin{tikzcd}
\check{H}^p(K;E_M) \arrow[r] \arrow[d, "\simeq"] & H_{n-p}(M, M\setminus K; E_M) \arrow[d, "\simeq"] \\
\check{H}^p(\{x\};E_M) \arrow[r, "\simeq"] & H_{n-p}(M, M\setminus \{x\}; E_M),
\end{tikzcd}$$

where the vertical maps are isomorphisms by the homotopy invariance and the bottom row by the observation above. Hence condition (1) is verified.

Condition (2) follows immediately from the five-lemma and Lemma B.10.

We turn to Condition (3). Let K_i be a sequence of compact subset with $K = \bigcap_{i\in\mathbb{N}} K_i$ and such that $P_M(K_i)$ holds for all $i \in \mathbb{N}$. It is an exercise in point-set topology of manifolds that each K_i has a fundamental system $U_{i,j}$ of open neighbourhoods. Fundamental system means that $U_{i,j} \subset U_{i,k}$ if $j < k$ and for each open set U containing K_i there is a j such that $U_{i,j} \subset U$. Another exercise in point-set topology of manifolds shows that one can construct these sets such that $U_{1,j} \supset U_{2,j} \supset U_{3,j} \supset \ldots$ for all $j \in \mathbb{N}$. Then $U_{i,j}$ is a fundamental system of open neighbourhoods of K with the order $(i,j) \leqslant (k,l)$ if and only if $i \leqslant k \wedge j \leqslant l$. One has the natural isomorphism [Br93, Appendix D5]:

$$\varinjlim_{i\in\mathbb{N}} \check{H}^p(K_i;E_M) = \varinjlim_{i\in\mathbb{N}} \varinjlim_{j\in\mathbb{N}} H^p(U_{i,j};E_M) \xrightarrow{\simeq} \varinjlim_{i,j\in\mathbb{N}} H^p(U_{i,j};E_M) \cong \check{H}^p(K;E_M).$$

And hence the theorem follows from the commutativity of the diagram:

$$\begin{array}{ccc}
\varinjlim_{i\in\mathbb{N}} \check{H}^p(K_i; E_M) & \longrightarrow & \varinjlim_{i\in\mathbb{N}} H_{n-p}(M, M\setminus K_i; E_M) \\
\downarrow & & \downarrow \\
\check{H}^p(K; E_M) & \longrightarrow & H_{n-p}(M, M\setminus K; E_M).
\end{array}$$

□

Duality with boundary

We obtain the following two observations. If M is closed, then we can take $K = M$ and obtain an isomorphism $H^p(M; E_M) \cong H_{n-p}(M; E_M)$, where the dualising map is given by capping with a generator $[M] \in H_n(M; \mathbb{Z}_M^{\mathrm{triv}})$. If M is compact with non-empty boundary one obtains another duality. Suppose that we have a collar $\partial M \times [0,2] \subset M$ of the boundary such that $\partial M = \partial M \times \{0\}$, then one has the following chain of isomorphisms:

$$\begin{aligned}
H^p(M; E_M) &\cong H^p(M \setminus (\partial M \times [0,1)); E_M) \quad \text{(homotopy)}\\
&\cong \check{H}^p(M \setminus (\partial M \times [0,1)); E_M) \quad \text{(Remark B.6)}\\
&\cong H_{n-p}(M \setminus \partial M, \partial M \times (0,1); E_M) \quad \text{(duality } K = M \setminus (\partial M \times [0,1)))\\
&\cong H_{n-p}(M, \partial M \times [0,1); E_M) \quad \text{(excision } U = \partial M)\\
&\cong H_{n-p}(M, \partial M; E_M).
\end{aligned}$$

It follows from the definition of the dualising map and naturality of the cap product that these isomorphisms are given by capping with a generator $[M, \partial M] \in H_n(M, \partial M; \mathbb{Z}^{\mathrm{triv}}) \cong H_n(M, \partial M; \mathbb{Z})$ as in the classical case. To get the other duality $H^p(M, \partial M; E_M) \cong H_{n-p}(M; E_M)$ one shows the commutativity of the following diagram and concludes with the five lemma (compare [Br93, Chapter VI Theorem 9.2]).

Theorem B.11. *The following diagram is commutative up to a sign:*

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & H^p(M; E_M) & \longrightarrow & H^p(\partial M; E_M) & \to & H^{p+1}(M, \partial M; E_M) & \longrightarrow & H^{p+1}(M; E_M) & \longrightarrow & \ldots \\
& & \downarrow{\scriptstyle \frown [M,\partial M]} & & \downarrow{\scriptstyle \frown [\partial M]} & & \downarrow{\scriptstyle \frown [M,\partial M]} & & \downarrow{\scriptstyle \frown [M,\partial M]} & & \\
\ldots & \to & H_{n-p}(M, \partial M; E_M) & \to & H_{n-p}(\partial M; E_M) & \to & H_{n-p-1}(M; E_M) & \to & H_{n-p-1}(M, \partial M; E_M) & \to & \ldots.
\end{array}$$

Proof . The commutativity is a more or less direct consequence of Equation (B.2) and the observation that $\partial_*[M, \partial M] = [\partial M]$. □

The mature reader may also want to prove the commutativity of the following diagram.

Theorem B.12. *Let* A *and* B *be codimension 0 submanifolds of* ∂M *such that* $\partial A = \partial B$ *and* $\partial M = A \cup B$. *The following diagram commutes up to a sign.*

$$\begin{tikzcd}
\dots \arrow[r] & H^p(M, A; E_M) \arrow[r] \arrow[dd, "\frown[M,\partial M]"] & H^p(M; E_M) \arrow[r] \arrow[dd, "\frown[\partial M]"] & H^{p+1}(A; E_M) \arrow[r] \arrow[d, "\frown[A]"] & H^{p+1}(M, A; E_M) \arrow[r] \arrow[dd, "\frown[M,\partial M]"] & \dots \\
& & & H_{n-p-1}(A, \partial A) \arrow[d] & & \\
\dots \arrow[r] & H_{n-p}(M, B; E_M) \arrow[r] & H_{n-p}(M, \partial M; E_M) \arrow[r] & H_{n-p-1}(A \cup B, B; E_M) \arrow[r] & H_{n-p-1}(M, B; E_M) \arrow[r] & \dots.
\end{tikzcd}$$

This gives the following duality theorem.

Theorem B.13. *Let* M *a compact oriented* n*-dimensional topological manifold. Let* A *and* B *be codimension 0 submanifolds of* ∂M *such that* $\partial A = \partial B$ *and* $\partial M = A \cup B$. *One has an isomorphism*

$$H^p(M, A; E_M) \cong H_{n-p}(M, B; E_M).$$

Bibliography

[Br93] G.E. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics, Springer New York (1993)

[HW08] G. Hardy and E. Wright, *Introduction to the theory of numbers*, Oxford University Press, 6 edition (2008)

[HS91] F. Hirzebruch and W. Scharlau, *Einführung in die Funktionalanalysis*, Spektrum Akademischer Verlag, (1991)